Friedrich-Alexander-University Erlangen-Nuremberg

Chair of Multimedia Communications and Signal Processing

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Master Thesis

Models for the Simulation of Dynamic Multidimensional Systems with Realistic Boundary Conditions: The Plate Equation

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FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG

Master thesis

for

Mr. cand. M.Sc. Manuel Werner

Models for the Simulation of Dynamic Multidimensional Systems with Realistic Boundary Conditions: The Plate Equation

Physical systems described by partial differential equations are dynamic multidimensional systems in terms of system theory. The development of effective simulation algorithms by a functional transformation approach is one of the research topics at the Chair of Multimedia Communications and Signal processing.

The correct consideration of boundary conditions is of high interest in the simulation of multidimensional systems. For scenarios in one spatial dimension it has been shown that complicated boundary conditions can be realized by feedback loops at a system boundary with simple boundary conditions. An extension to two spatial coordinates is of special interest for the plate equation, since analytical solutions exist only for the most simple boundary conditions. Such an approach can be used in practice for physical modelling of musical instruments.

The task to develop a simulation model for the plate equation with simple boundary conditions based on functional transformations is assigned to Mr. Manuel Werner. The derived model shall be extendable to more complex boundary conditions.

Of high importance is a detailed documentation of the developed program code and a report which describes the theory, the realization and the results.

Beginn: 04.01.2017
Ende: 21.07.2017

R. Rabenstein

(Prof. Dr.-Ing. R. Rabenstein)

Erklärung

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Contents

Abstract

This thesis deals with the physical modelling of vibrating plates that are commonly represented by a corresponding partial differential equation. First the physical foundation of a vibrating plate is introduced, which is then constituted by a representation of a partial differential equation. Furthermore, analytical solutions for the dynamic behaviour of plates are demonstrated and investigated. Second the theoretical concept and the application of the functional transformation method are shown, since this method serves many modelling advantages. Last the functional transformation method is applied to the specific two-dimensional plate equation, whereas its solution is compared to the Navier's solution as a reference. This thesis states the basis for the application of adjustable boundary conditions and the digital sound synthesis of musical instruments like the xylophone and marimbaphon.

Index of Abbreviations

w.r.t.	with regard to
FTM	Functional Transformation Model
PDE	Partial Differential Equation
SLT	Sturm-Liouville Transformation
Eq.	Equation
Fig.	Figure
Chap.	Chapter
Sec.	Section

Mathematical Symbols

$+$	Addition
$-$	Subtraction
$\boldsymbol{A}$	Matrix
$\boldsymbol{y}(\boldsymbol{x}, t)$	Vector w.r.t space and time
$\boldsymbol{Y}(\boldsymbol{x}, s)$	Laplace Transformation of $\boldsymbol{y}(\boldsymbol{x}, t)$
L	Differential Operator
$\tilde{L}$	Adjoint Operator
$\langle \boldsymbol{u}, \boldsymbol{v} \rangle$	Scalar Product of vector $\boldsymbol{u}$ and $\boldsymbol{v}$

Chapter 1

Introduction

There are many ways to create sound. Handmade musical instruments are the origin of a controlled sound production and are still present in many musical styles. Furthermore, by means of nowadays electrical components and digital signal processing new musical instruments were developed and digital sound synthesis emerged. Digital sound synthesis is engrained in the modern music production and many different methods were developed to generate and recreate sound. In general, the following methods of sound synthesis are known: Wavetable Synthesis, Granular Synthesis, Additive and Subtractive Synthesis, Frequency Modulation Synthesis and Physical Modeling [1][2]. When sound of musical instruments shall be synthesized the method of physical modeling plays an important role in the digital signal processing, since it aims for an exact physical representation of the instrument and its resulting wave generation [3][4].
Musical instruments generate sound and specific waveforms that underlie a certain physical principle. The movements of material – like the oscillating string of a guitar – result in sound that is perceived over time. Moreover, its sound characteristics are defined by physical constraints and conditions. Technically spoken, the sound generation of such physical systems with time and space as independent variables can often be described by partial differential equations [5].
The physical modeling of musical instruments exploits exactly those physical properties and represents the sound behaviour in a mathematical model. The Functional

Transformation Method (FTM) describes a defined approach in order to find a discrete synthesis algorithm of musical instruments for real-time applications. Transfer function models for one-dimensional musical instruments, like the guitar string was investigated for instance in [3]. The physical modeling of two-dimensional instruments is less studied for cases where the mathematical description is not separable in two single dimensions. Therefore, this thesis deals with the physical modeling of two-dimensional plates for simple boundary conditions and states the basis for the application of adjustable boundary conditions. Results of this thesis can be used for the digital sound synthesis of musical instruments like for instance the marimbaphon or xylophone.

At first the general plate physics can be found in Chapter 2 and analytical solutions for the vibrating plate are explained. In Chapter 3 the theory of the Functional Transformation Method is described by introducing the Laplace- and Sturm-Liouville Transformation. Based on the theory of vibrating plates the FTM is applied to the plate equation in Chapter 4. Finally, the resulting solution of FTM for thin plates is compared to analytical solutions.

Chapter 2

Plate Physics

In this chapter the physics of a vibrating plate are stepwise developed. Therefore, the geometry and elasticity of the plate are investigated and meaningful relations of stresses and strains are defined. The aim of this section is to establish a partial differential equation (PDE) that represents the static and dynamic behaviour of a plate. For further calculations in the next sections a solid formulation of boundary conditions is required and shown for an analytical solution. In Sec. 2.1 the static physical behaviour of a plate is explicitly shown. Furthermore, by the introduction of time-dependent motion the static description is extended to a dynamic one. In the end Sec. 2.2 offers general analytical solutions for the partial differential equation that can be used for later comparisons.

2.1 Physical Properties of a Plate

The Figure 2.1 depicts the geometry of a rectangular plate that can be generally determined by the lengths L_x, L_y and the height h. With regard to this geometry the physical behaviour is investigated in more detail. Two general situations can be described when forces acting upon a deformable body: First the influence of stresses and second its result in form of strains (body after deformation). Stresses are represented basically by forces that are applied on a volume or on an area as the surface of a body

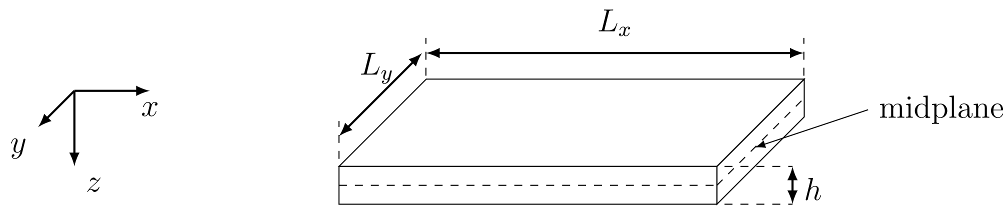


Figure 2.1: Three-dimensional plate with the lengths L_x, L_y and the height h.

[6, p.3-4]. The so-called stress resultants define further forces like bending, twisting moments and vertical shear forces. When considering deformable bodies strains occur that are incorporated by extensions or contractions of the material [7, p.5]. Depending on the condition of equilibrium the body is either just deformed or/and moved. The two concepts of stresses and strains are introduced in the following chapter and then combined by the linear elasticity of Hooke's Law. For the definition of stresses and strains the concept of equilibrium is necessary and simplifications of the behaviour of a plate are made according to Kirchhoff. Based on this, further Strain-Curvature relations are introduced and the partial differential equation for a bending plate is derived. With regard to the PDE a description of possible boundary conditions is of particular interest and needed for further calculations.

2.1.1 Kirchhoff Hypotheses

For the subsequent formulation of a plate theory it seems to be beneficial to introduce some fundamental assumptions in the beginning. For this reason the following Kirchhoff hypotheses are mentioned, since they show quite many, but reasonable simplifications for the plate-bending theory of thin plates [8, p.72-73][6, p.100-102]:

1. The midplane remains unstrained subsequent to bending. The normals to the midsurface remain straight and do not change their perpendicular orientation to the deformed midsurface. The slope is relatively small and leads to the strain-curvature relations in Sec. 2.1.5.

2. The displacement w of a point is independent of z ($w = w(x, y)$). Accordingly,

all points on the normal are displaced equally and the thickness of the plate remains unchanged (See Fig. 2.5, point B). Therefore, the normal strain ϵ_z from the transverse loading may also be omitted.

3. The vertical shear strains along the sides of a plate (γ_{xz} and γ_{yz}) are negligible when considering twisting moments ($M^{(xz)}, M^{(yz)}$). However, for the development of vertical forces ($Q^{(x)}, Q^{(y)}$) shear stresses τ_{xz} and τ_{yz} are relevant.

4. The stress normal to the midplane is small compared with the other stress components and may be neglected: $\sigma_z \ll \sigma_x, \sigma_y, \tau_{xy}$. The deflection of the plate is thus associated principally with bending strains.

These assumptions require the idea of thin plates and relatively small deflections. Due to the mentioned decrease of complexity a three-dimensional problem is hereby reduced to an basically two-dimensional problem. In the following sections these assumptions are applied in order to derive a partial differential equation in a concise manner.

2.1.2 Stresses

First of all stresses must be defined for certain points in a body in order to describe deformations or movements. Therefore, it is beneficial to consider an infinitesimal element of a body. Figure 2.2 shows such an element and all possible locations of stresses acting upon a body. Depending on its direction these stresses are called normal (σ) or shear stresses (τ) [8, p.6-8, p.76-79]. In general, a sign convention can be defined as follows. If a stress component acts on a positive face in a positive coordinate direction, the stress component is positive. So both the face and the coordinate direction have to be positive for a positive stress. When in a positive face the coordinate direction is negative the stress is considered as negative and vice versa.

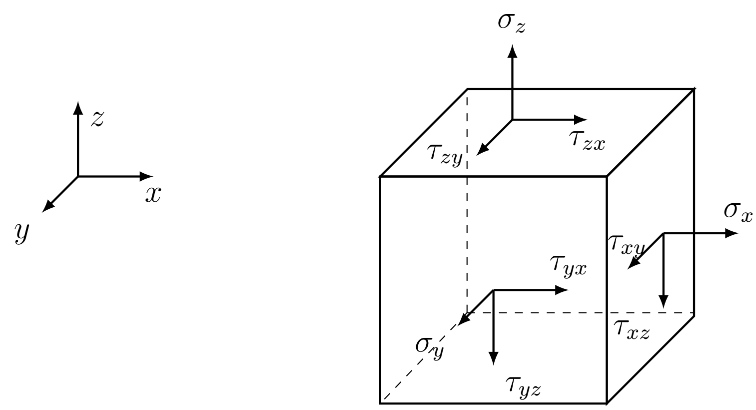


Figure 2.2: Positive stresses shown in a three-dimensional element. Normal stresses defined as σ and shear stresses as τ.

These stresses can be assembled in form of a stress tensor

$$[\tau_{ij}] = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}. \tag{2.1}$$

The first subscript of τ and σ denotes the perpendicular direction of the plane on which the stress acts. The second subscript denotes the direction of the impact. Here, it is necessary to mention that a certain symmetry exists and the following connection between stresses is established

$$\tau_{xy} = \tau_{yx}, \qquad \tau_{zy} = \tau_{yz}, \qquad \tau_{xz} = \tau_{zx}. \tag{2.2}$$

Intuitively, these shear stresses can be interpreted as a resulting rotation of the body, where it does not matter on which side of the "cube" the force is acting. This consideration justifies the definition of stresses of Eq. (2.2).

Moreover, moments can be established that are derived by the stresses in Eq. (2.1). These moments are measured over the thickness of the plate and can be categorised in resulting bending, twisting moments and vertical shear forces. The resultants of bending moments are calculated w.r.t the Kirchhoff hypotheses as

$$\int_{-h/2}^{h/2} z\sigma_x \mathrm{d}z\mathrm{d}y = \mathrm{d}y \int_{-h/2}^{h/2} z\sigma_x \mathrm{d}z = M^{(x)}\mathrm{d}y, \tag{2.3}$$

$$M^{(x)} = \int_{-h/2}^{h/2} \sigma_x z \mathrm{d}z, \tag{2.4}$$

$$M^{(y)} = \int_{-h/2}^{h/2} \sigma_y z \mathrm{d}z. \tag{2.5}$$

Note that $M^{(z)} = 0$ due to $\sigma_z = 0$ (see the fourth Kirchhoff's assumption in Sec. 2.1.1). Bending moments are denoted by a single superscript ($M^{(x)}$, $M^{(y)}$), whereas the twisting moments are denoted by two superscripts ($M^{(xy)}$). The twisting moment can be expressed as

$$M^{(xy)} = \int_{-h/2}^{h/2} \tau_{xy} z \mathrm{d}z, \tag{2.6}$$

where $M^{(xy)} = M^{(yx)}$ due to the ideal symmetry behaviour [8, p.8]. Analogously, the vertical shear forces are

$$Q^{(x)} = \int_{-h/2}^{h/2} \tau_{xz} dz, \tag{2.7}$$

$$Q^{(y)} = \int_{-h/2}^{h/2} \tau_{yz} dz. \tag{2.8}$$

Due to the integration over the thickness of the plate the vertical shear forces and moments have the dimensions of force per unit length.

2.1.3 Strains

Strains can be seen as deformations of a body. In order to define those deformations, deflections in the direction of the three dimensions must be stated. These can be distinguished between linear and shear strains depending on the direction of strains. The deflection in the x-direction is described by the parameter v and in the y-direction by the paramenter u. Accordingly, the movement in z-direction is defined by the paramenter w. Figures 2.3 and 2.4 show possible deformations caused by linear and shear strains in the xy-plane [6, p.23].

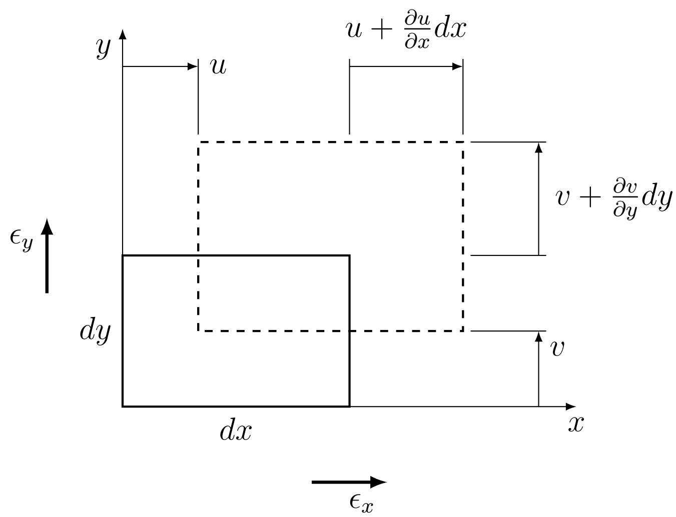


Figure 2.3: Straight drawing caused by linear strains ϵ_x and ϵ_y in the two-dimensional xy-plane.

Contrarily to linear strains, the shear strains represent the dependency of more directions that visually can be seen as a diagonal strain. Additionally to the linear deformation in x- or y-direction, the in- or decrease of u and v depends also on the corresponding perpendicular direction. Using the derivative notation the linear strains are defined as [9, p.3]

$$\epsilon_x = \frac{\partial u}{\partial x}, \tag{2.9}$$

$$\epsilon_y = \frac{\partial v}{\partial y}. \tag{2.10}$$

And further the shear strain is defined as

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{2.11}$$

Applying the last two Kirchhoff hypotheses, the linear and shear strains of the plate in z-direction are neglected (for later twisting moments): $\epsilon_z = 0$, $\gamma_{xz} = 0$ and $\gamma_{yz} = 0$.

2.1.4 Hooke's Law and Poisson's Rate

Hooke's law defines a linear relation between stress and strain and introduces the constant modulus of elasticity or Young's modulus E [9, p.6-7]. This relation is valid

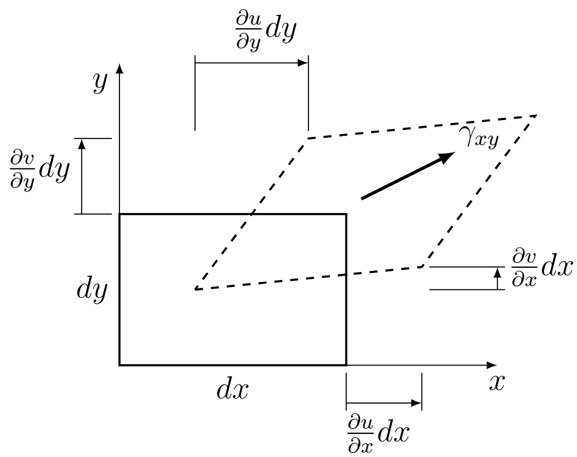


Figure 2.4: Diagonal drawing caused by shear strain γ_{xy} in the two-dimensional xy-plane.

for the elastic range and does not hold for very strong stresses and strains

$$\sigma = E\epsilon. \tag{2.12}$$

In a similar way the modulus of rigidity G is defined for shear loading

$$\tau = G\gamma. \tag{2.13}$$

As a further step the Poisson's rate shows which element deformations are caused by biaxial stresses. Poisson's rate is defined as

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}}. \tag{2.14}$$

Inserting Eq. (2.14) into (2.12) the following equations can be derived

$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}, \qquad \epsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E}. \tag{2.15}$$

These equations can be reformulated to achieve a representation for the according stresses

$$\sigma_x = \frac{E}{1-\nu^2}\left(\epsilon_x + \nu\epsilon_y\right), \qquad \sigma_y = \frac{E}{1-\nu^2}\left(\epsilon_y + \nu\epsilon_x\right). \tag{2.16}$$

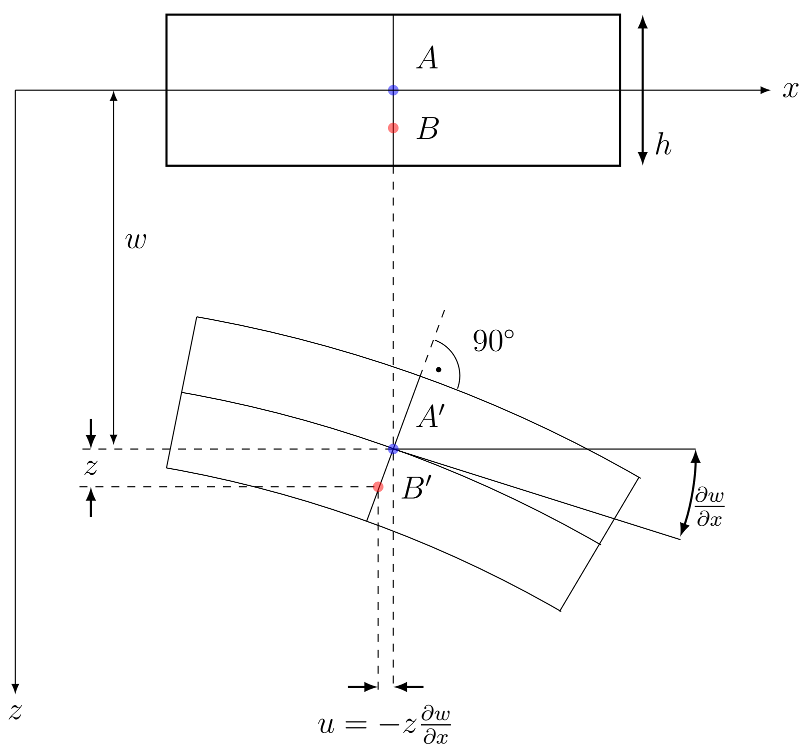


Figure 2.5: Shift in x- and z-direction of points before and after deflection. Point A' is located on the midplane contrarily to point B' that is shifted in x- and z-direction.

2.1.5 Strain-Curvature Relations

This section discusses the geometry of possible deflections in detail. Fig. 2.5 shows the influence of deflection in the z-direction [10, p.277]. Note that under this impact the point B (that is not located on the midplane) is shifted in x- and z-direction. In order to express this two-dimensional shift the angle of the tangent in point B' and the distance z are relevant. The shift in x-direction is expressed as

$$u = -z\frac{\partial w}{\partial x}. \tag{2.17}$$

Analogously, the shift in y-direction follows as

$$v = -z\frac{\partial w}{\partial y}. \tag{2.18}$$

Inserting Eqs. (2.17) and (2.18) in Eqs. (2.9) - (2.11) leads to the following equations for linear and shear strains

$$\epsilon_x = -z\frac{\partial^2 w}{\partial x^2}, \tag{2.19}$$

$$\epsilon_y = -z\frac{\partial^2 w}{\partial y^2}, \tag{2.20}$$

$$\gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y}. \tag{2.21}$$

Now the stress-strain relation from (2.16) can be expressed as

$$\sigma_x = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \tag{2.22}$$

$$\sigma_y = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right). \tag{2.23}$$

Inserting Eq. (2.21) into Eq. (2.13) leads to a representation of the shear load in terms of deflection ω

$$\tau_{xy} = -\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x \partial y}, \tag{2.24}$$

with the modulus of ridigity defined as

$$G = \frac{E}{2(1+\nu)}. \tag{2.25}$$

2.1.6 Conditions of Equilibrium

For a first consideration of all forces acting on a body the condition of equilibrium is applied. The conditions of equilibrium states that a composition of forces acting upon a plate does not result in movements. The load-carrying system is in balance. This can be mathematically expressed in terms of forces in xyz-direction [8, p.6]

$$\sum F_x = 0, \qquad \sum F_y = 0, \qquad \sum F_z = 0. \tag{2.26}$$

2.1.7 Static Partial Differential Equation

Based on the previous sections the resultants of bending and twisting moments can be calculated and the differential equation can be stated. Inserting Eqs. (2.22)-(2.24) into

Eqs. (2.4)-(2.6) leads to a beneficial representation of the moments

$$M^{(x)} = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \tag{2.27}$$

$$M^{(y)} = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right), \tag{2.28}$$

$$M^{(xy)} = -D\left(1-\nu\right)\frac{\partial^2 w}{\partial x \partial y}, \tag{2.29}$$

in terms of deflection and the flexural rigidity $D = \frac{Eh^3}{12(1-\nu^2)}$ [9, p.12]. Analogously, the shear forces can be expressed in terms of w and D

$$Q^{(x)} = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right), \tag{2.30}$$

$$Q^{(y)} = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x^2}\right). \tag{2.31}$$

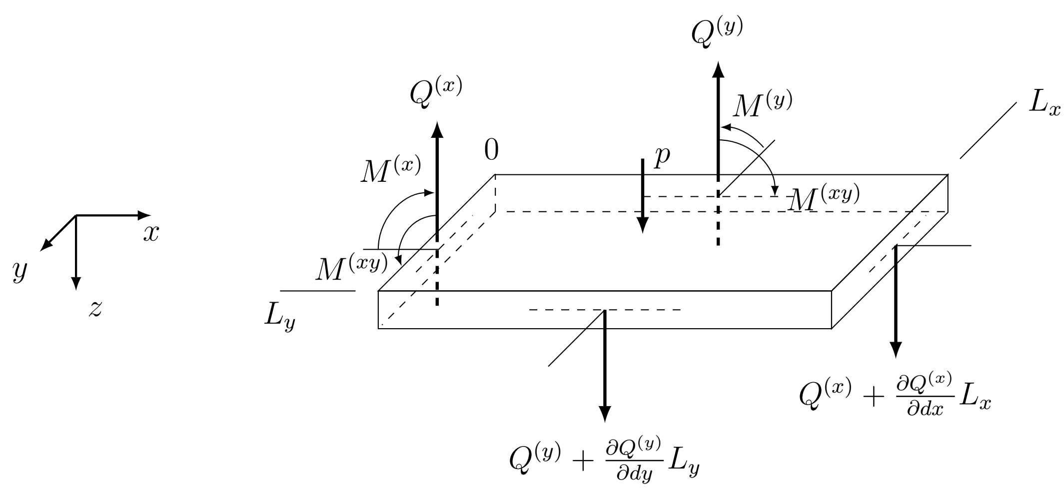


Figure 2.6: The distribution of moments and shear forces over the plate and that directional effect dependent on the specific location in the xy-plane.

The partial differential equation is developed by the conditions of equilibrium of Eq. (2.26). Therefore, the sum of the forces in each direction must be zero (see Sec. 2.1.6). Figure 2.6 shows a plate with the lengths of L_x, L_y and the height h [8, p.81]. For simplicity the potential stresses are shown at $x = 0$ and $y = 0$. However, these stresses are spread over the entire plate, where for instance a change of location in x-direction can be expressed by the truncated Taylor's expansion

$$Q^{(x)} + \frac{\partial Q^{(x)}}{\partial x} L_x. \tag{2.32}$$

Considering stresses acting on the entire plate the area in x-direction and y-direction has to be taken into account. Since the state of equilibrium (with regard to the dimensions of the plate) is assumed, the difference of stresses at different locations is necessary. Based on that consideration the equilibrium in z-direction can be developed using the truncated Taylor's expansion

$$\frac{\partial Q^{(x)}}{\partial x} L_x L_y + \frac{\partial Q^{(y)}}{\partial y} L_x L_y + p L_x L_y = 0, \tag{2.33}$$

$$\frac{\partial Q^{(x)}}{\partial x} + \frac{\partial Q^{(y)}}{\partial y} + p = 0, \tag{2.34}$$

whereas p describes the load acting on the plate. In x-direction subsequent equations are derived

$$\frac{\partial M^{(xy)}}{\partial x} L_x L_y + \frac{\partial M^{(y)}}{\partial y} L_x L_y - Q^{(y)} L_x L_y = 0, \tag{2.35}$$

$$\frac{\partial M^{(xy)}}{\partial x} + \frac{\partial M^{(y)}}{\partial y} - Q^{(y)} = 0. \tag{2.36}$$

And in y-direction it can be shown that

$$\frac{\partial M^{(xy)}}{\partial x} L_x L_y + \frac{\partial M^{(x)}}{\partial y} L_x L_y - Q^{(x)} L_x L_y = 0, \tag{2.37}$$

$$\frac{\partial M^{(xy)}}{\partial y} + \frac{\partial M^{(x)}}{\partial x} - Q^{(x)} = 0. \tag{2.38}$$

Inserting Eqs. (2.36) and (2.38) in Eq. (2.34) yields to the differential equation of equilibrium

$$\frac{\partial^2 M^{(x)}}{\partial x^2} + 2\frac{\partial^2 M^{(xy)}}{\partial x \partial y} + \frac{\partial^2 M^{(y)}}{\partial y^2} = -p. \tag{2.39}$$

Reusing the Eqs. (2.27)-(2.29) and applying them to Eq. (2.39) leads to the differential equation of deflection

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}. \tag{2.40}$$

Collecting the derivatives into a Laplace operator, the partial differential equation is rewritten as

$$\Delta\Delta w = \frac{p}{D}. \tag{2.41}$$

If no lateral load is acting Eq. (2.41) simplifies to

$$\Delta\Delta w = 0 \tag{2.42}$$

with $\Delta\Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2$.

2.1.8 Dynamic Partial Differential Equation

If the equilibrium of equation (2.41) is not fulfilled, the problem can not be solved by a formulation of stresses and strains of the body. Thus, the approach from the previous section is extended to a dynamic reformulation in this section.
As a consequence it can be stated that the body is accelerated to a certain direction [6, p.14-15]. In Sec. 2.1.7 is shown that the following moments are missing and thus can not lead to movements in x- and y-direction

$$M_{xz} = \int_{-h/2}^{h/2} \tau_{xz} z dz = 0 \qquad (\tau_{xz} = G\gamma_{xz} = 0), \tag{2.43}$$

$$M_{yz} = \int_{-h/2}^{h/2} \tau_{yz} z \mathrm{d}z = 0 \qquad (\tau_{yz} = G\gamma_{yz} = 0). \tag{2.44}$$

Under the consideration of Kirchhoff's hypothesis the body can only accelerate in the z-direction. With regard to Newton's second law [11, p.106] the resulting force can be represented by a mass multiplied by its acceleration. Thus, the resulting force is derived from Eq. (2.33)

$$\begin{aligned}
\frac{\partial Q^{(x)}}{\partial x} L_x L_y + \frac{\partial Q^{(y)}}{\partial y} L_x L_y + p L_x L_y &= (\rho V)\,\frac{\partial^2 w}{\partial t^2}, \\
\frac{\partial Q^{(x)}}{\partial x} + \frac{\partial Q^{(y)}}{\partial y} + p &= \left(\rho \frac{V}{L_x L_y}\right) \frac{\partial^2 w}{\partial t^2}, \\
\frac{\partial Q^{(x)}}{\partial x} + \frac{\partial Q^{(y)}}{\partial y} + p &= (\rho h)\,\frac{\partial^2 w}{\partial t^2}.
\end{aligned} \tag{2.45}$$

A proof of dimensions is shown in the appendix A.1. The mass of the body depends on the height h and the density of the body ρ. The dependency of the height h is derived from the consideration that $h = \frac{V}{L_x L_y}$ and refers to the thickness of the plate. Inserting Eqs. (2.36) and (2.38) leads to

$$\frac{\partial^2 M^{(x)}}{\partial x^2} + 2\frac{\partial^2 M^{(xy)}}{\partial x \partial y} + \frac{\partial^2 M^{(y)}}{\partial y^2} = -p + (\rho h)\frac{\partial^2 w}{\partial t^2}, \tag{2.46}$$

$$-D\left[\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] = -p + (\rho h)\frac{\partial^2 w}{\partial t^2}, \tag{2.47}$$

$$\Delta\Delta w = \frac{p}{D} - \frac{1}{D}(\rho h)\frac{\partial^2 w}{\partial t^2}, \tag{2.48}$$

$$\Delta\Delta w - \frac{p}{D} + \frac{\rho h}{D}\frac{\partial^2 w}{\partial t^2} = 0. \tag{2.49}$$

Introducing an excitation function $f_\text{e} = \frac{p}{D}$, Eq. (2.49) is reformulated to the PDE

$$\Delta\Delta w + R\frac{\partial^2 w}{\partial t^2} = f_\text{e}, \tag{2.50}$$

with $R = \frac{\rho h}{D}$ and the flexural rigidity $D = \frac{Eh^3}{12(1-\nu^2)}$. The developed equation is further known as the Kirchhoff thin plate model [12, p.331].

2.1.9 Boundary Conditions

In order to solve the given PDE of Eq. (2.50) boundary conditions are necessary. In general, boundary conditions describe the behaviour at the edges of an object or system, for instance a fixed position, slope or twisting moment. They depend thus on the coordinates x and y and its derivatives. The most common boundary conditions are fixed and free edge, simply supported edge and sliding edge, whereas each of them is represented by two conditions [10, p.286][6, p.106]. The PDE of Eq. (2.50) requires two conditions for each edge, so in total eight conditions. Before introducing the specific boundary conditions the edge effect of the twisting moment must be explained. Here, it is necessary to apply an effective transverse force that is calculated from the

Eqs. (2.27)-(2.31)

$$V^{(x)} = Q^{(x)} + \frac{\partial M^{(xy)}}{\partial y} = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right], \tag{2.51}$$

$$V^{(y)} = Q^{(y)} + \frac{\partial M^{(xy)}}{\partial x} = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right]. \tag{2.52}$$

The terms $\frac{\partial M^{(xy)}}{\partial y}$ and $\frac{\partial M^{(xy)}}{\partial x}$ are developed by an equivalent replacement of the twisting moments at the edges. For an infinitesimal small deflection dy just the increase of $\frac{\partial M^{(xy)}}{\partial y}dy$ together with $Q^{(x)}$ or $Q^{(y)}$ represent the effective force in z-direction.

Due to the symmetry the subsequent boundary conditions hold for the x- and y-direction. However, they are just mentioned for the x-direction with regard to reasons of brevity. Substituting x by y leads to the according boundary conditions for the y-direction.

Fixed Edge

In the case of a fixed edge the deflection and the slope must vanish (see Fig. 2.7). This can be expressed by boundary conditions for the deflection and its first space derivative

$$w = 0 \qquad (x = 0 \vee x = L_x), \tag{2.53}$$

$$\frac{\partial w}{\partial x} = 0 \qquad (x = 0 \vee x = L_x). \tag{2.54}$$

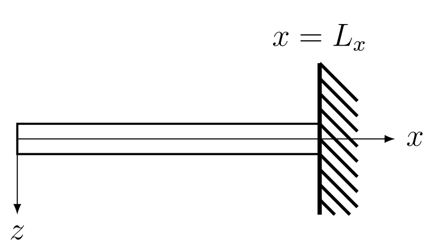


Figure 2.7: Boundary condition of a fixed edge at $x = L_x$ requires that the deflection and slope must vanish.

Free End

Another possible boundary condition is a free end of the plate. Here, the bending moments and the effective transverse force play an important role

$$M^{(x)} = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) = 0 \qquad (x = 0 \vee x = L_x), \tag{2.55}$$
$$V^{(x)} = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right] = 0 \qquad (x = 0 \vee x = L_x). \tag{2.56}$$

Simply Supported Edge

In the case of simply supported edge the deflection and the bending moment are set to zero at the boundaries (see Fig. 2.8)

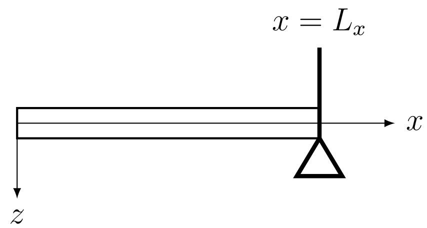
$$w = 0 \qquad (x = 0 \vee x = L_x), \tag{2.57}$$
$$\frac{\partial^2 w}{\partial x^2} = w_{xx} = 0 \qquad (x = 0 \vee x = L_x). \tag{2.58}$$

Furthermore, the bending moment can be expressed as

$$M^{(x)} = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) = 0 \qquad (x = 0 \vee x = L_x), \tag{2.59}$$

whereas along the edge ($x = L_x$) the first derivative in y-direction vanishes $\frac{\partial w}{\partial y} = w_y = 0$ and thus $\frac{\partial^2 w}{\partial y^2} = w_{yy} = 0$. This consideration can be reused again for the bending moment in the y-direction

$$M^{(y)} = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) = 0 \qquad (x = 0 \vee x = L_x). \tag{2.60}$$



Boundary Conditions:

$$w(L_x, y) = 0$$
$$M^{(x)}(L_x, y) = 0$$
$$M^{(y)}(L_x, y) = 0$$

Figure 2.8: Boundary condition of a simply supported edge at $x = L_x$ requires that the deflection and bending moment are set to zero.

Sliding Edge

If the boundary condition of a sliding edge is applied, the edge is just able to move vertically and no rotation is possible (no vertical shear forces, see Fig. 2.9) [8, p.86]

$$\frac{\partial w}{\partial x} = 0 \qquad (x = 0 \vee x = L_x), \tag{2.61}$$

$$V^{(x)} = -D \left[\frac{\partial^3 w}{\partial x^3} + (2 - \nu) \frac{\partial^3 w}{\partial x \partial y^2} \right] = 0 \qquad (x = 0 \vee x = L_x). \tag{2.62}$$

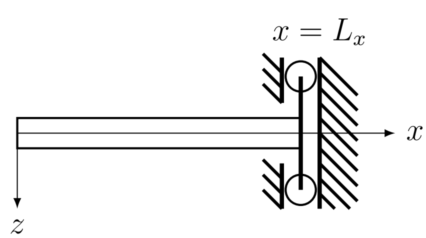


Figure 2.9: Boundary condition for sliding edge at $x = L_x$ requires just vertical movements and no rotations.

2.2 Analytical Solutions for the Partial Differential Equation of Plates

In this section analytical solutions for the PDE of Eq. (2.50) are developed. The first procedure that is applied to the PDE is stating a fundamental solution. The boundary conditions of simply supported edges are therefore used. The result is afterwards proofed by reinserting the solution in the given PDE. Another approach is known as the Naviers's solution that is also elaborated in that section.

2.2.1 Dynamic Solution for the Vibrating Plate

In this section analytical solutions for the derived PDE of Eq. (2.50) are developed. One way is to use a fundamental and general solution that is inserted into the PDE. The main goal is to solve the PDE by applying the boundary conditions stepwise. The following procedure shows a possible way for a simply supported plate.

First of all, general calculations for later use are made. For $R = 1$ and $f_\mathrm{e} = 0$ the PDE (2.50) can expressed as [13, p.307]

$$\Delta\Delta w + w_{tt} = 0. \tag{2.63}$$

The Fourier-Transformation is defined as

$$W(j\omega) = \mathcal{F}\left\{w(t)\right\} = \int_0^\infty w(t)e^{-j\omega t}dt, \tag{2.64}$$

with the differentiation theorem

$$\frac{\partial^2}{\partial t^2}\left(e^{-j\omega t}\right) = -\,\omega^2 \cdot e^{-j\omega t}. \tag{2.65}$$

When applying the Fourier-Transformation of Eq. (2.64) and its differentiation theorem of Eq. (2.65) the PDE in Eq. (2.63) can be reformulated as

$$\Delta\Delta W(x,y,\omega) - \omega^2 W(x,y,\omega) = 0, \tag{2.66}$$

which can be rewritten to

$$\Delta\Delta W(x,y,\omega) - k_0^4 W(x,y,\omega) = 0, \tag{2.67}$$

with $k_0^4 = \omega^2$. With regard to the Sec. 2.1.9 two boundary conditions for each edge can be set. In the case of four simply supported edges eight boundary conditions can

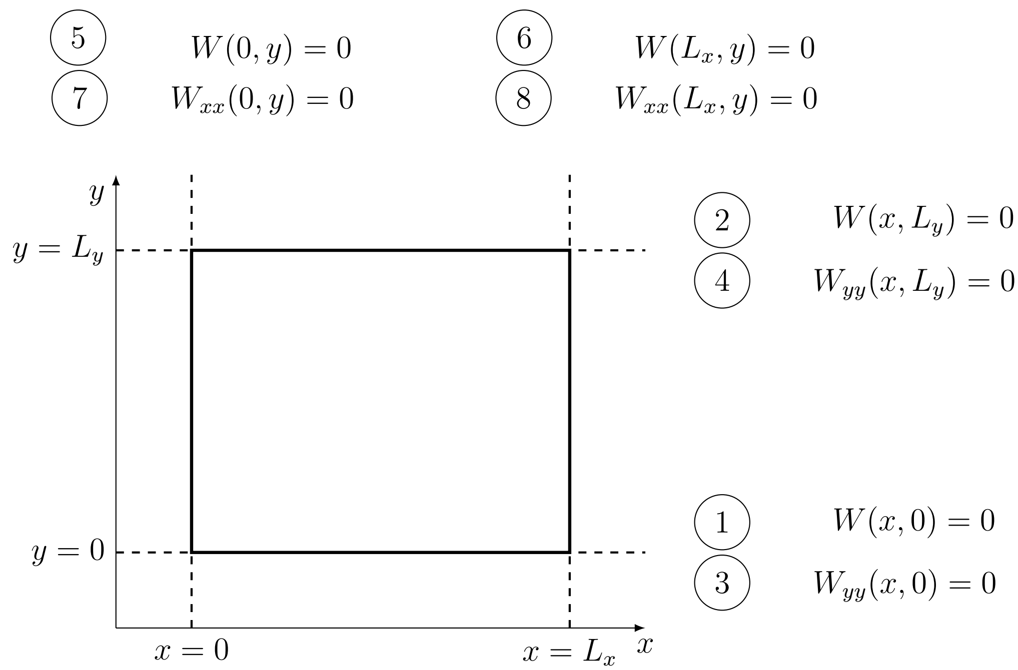


Figure 2.10: Boundary conditions for simply supported edges.

be stated in the following way (see Fig. 2.10). The fundamental solution for the PDE in Eq. (2.67) is defined as

$$W(x,y)=\hat{W}\cdot e^{jk_xx+jk_yy}=\hat{W}\cdot e^{jk_xx}\cdot e^{jk_yy}, \tag{2.68}$$

whereas k_x, k_y can be real- and complex-valued. The Laplace-Operator is known as

$$\Delta=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right), \tag{2.69}$$

$$\Delta\Delta=\left(\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right). \tag{2.70}$$

The partial derivatives of W (see Eq. (2.68)) are calculated as

$$\frac{\partial^2}{\partial x^2}W(x,y)=-k_x^2W(x,y), \qquad \frac{\partial^2}{\partial y^2}W(x,y)=-k_y^2W(x,y), \tag{2.71}$$

$$\frac{\partial^4}{\partial x^4}W(x,y)=k_x^4W(x,y), \qquad \frac{\partial^4}{\partial y^4}W(x,y)=k_y^4W(x,y), \tag{2.72}$$

and

$$\frac{\partial^4}{\partial x^2\partial y^2}W(x,y)=k_x^2k_y^2W(x,y). \tag{2.73}$$

Inserting the fundamental solution of Eq. (2.68) and its derivatives of Eqs. (2.71)-(2.73) into the PDE (2.67) leads to

$$\left(k_x^4 + 2k_x^2 k_y^2 + k_y^4\right) W(x,y) - k_0^4 W(x,y) = 0. \tag{2.74}$$

Based on that result the dispersion relation can be established as

$$\Rightarrow \left(k_x^2 + k_y^2\right)^2 = k_0^4 \tag{2.75}$$

that defines the connection of spatial and temporal frequencies. From now on a case analysis is necessary. The spatial frequencies have to be determined with regard to real- and complex-valued components. The subsequent Fig. 2.11 shows all possible configurations of the spatial frequencies, whereas λ is considered as a real number.

k_y \ k_x	real-valued	complex-valued
real-valued	**4** $k_x = \lambda_x$ $k_y = \lambda_y$	**2** $k_x = j\lambda_x$ $k_y = \lambda_y$
complex-valued	**3** $k_x = \lambda_x$ $k_y = j\lambda_y$	**1** $k_x = j\lambda_x$ $k_y = j\lambda_y$

Figure 2.11: Case analysis: Possible configurations of real- and complex-valued frequencies k_x and k_y.

First Case: Imaginary Spatial Frequencies

In the first case both spatial frequencies k_x and k_y are complex-valued

$$k_x^2 = -\lambda_x^2, \qquad k_y^2 = -\lambda_y^2. \tag{2.76}$$

Under this assumption the Eq. (2.75) can be further reformulated

$$
\begin{aligned}
(-\lambda_x^2 - \lambda_y^2)^2 &= k_0^4, \\
(\lambda_x^2 + \lambda_y^2)^2 &= k_0^4, \\
-\lambda_x^2 - \lambda_y^2 &= k_0^2, \qquad \text{due to} \qquad \lambda_x, \lambda_y \in \mathbb{R} &(2.77)\\
\lambda_y^2 &= k_0^2 - \lambda_x^2 = k_0^2 + k_x^2, &(2.78)\\
\lambda_y &= \pm\sqrt{k_0^2 + k_x^2} = \pm\lambda_{y0}. &(2.79)
\end{aligned}
$$

From that result it can be seen that λ_{y0} is a positive number, so that

$$\lambda_{y0} = \sqrt{k_0^2 + k_x^2} \geq 0. \tag{2.80}$$

With regard to two solutions of λ_y in Eq. (2.79) the fundamental solution is expanded. The subscript of W_{12} denotes the dependency of both solutions. The deflection $W(x,y)$ is defined as the sum of both solutions (see Eq. (2.82)). Applying Eq. (2.79) to Eq. (2.68) leads to

$$W_{12}(x,y) = \hat{W} e^{-\lambda_x x} e^{\pm\lambda_{y0} y}, \tag{2.81}$$

$$W(x,y) = \sum_{n=1}^{2} W_n(x,y), \tag{2.82}$$

whereas the individual solutions are known as

$$W_1(x,y) = \hat{W}_1 e^{-\lambda_x x} e^{\lambda_{y0} y}, \tag{2.83}$$

$$W_2(x,y) = \hat{W}_2 e^{-\lambda_x x} e^{-\lambda_{y0} y}. \tag{2.84}$$

Inserting the **first boundary condition** $W(x,y) = 0$ for $y = 0$ leads to

$$\hat{W}_1 = -\hat{W}_2 = \frac{\hat{W}}{2}, \tag{2.85}$$

$$W(x,y) = \frac{\hat{W}}{2} e^{-\lambda_x x} \left(e^{\lambda_{y0} y} - e^{-\lambda_{y0} y}\right), \tag{2.86}$$

$$W(x,y) = \hat{W} e^{-\lambda_x x} \sinh(\lambda_{y0} y). \tag{2.87}$$

The **second boundary condition** $W(x,y) = 0$ for $y = L_y$ results in the following equation

$$W(x,y) = \hat{W}e^{-\lambda_x x}\sinh(\lambda_{y0}L_y) = 0. \tag{2.88}$$

Having in mind that λ_{y0} is set to real-valued the entire argument of sinh is real-valued. Thus, the only solution is $\lambda_{y0} = 0$. However, based on this condition there is no spatial freqency in y-direction at all. If $\lambda_{y0} \neq 0$, the second boundary condition is not fulfilled and thus does not lead to a solution for the PDE of (2.68). Furthermore, due to the symmetric solution of k_x and k_y this consequence holds also in the x-direction. Hence, there is no solution for complex-valued k_x and k_y for the given PDE.

Second Case: Imaginary- and Real-valued Spatial Frequencies

With regard to the Fig. 2.11, Case 2 describes the following combination of frequencies: $k_x = j\lambda_x$ and $k_y = \lambda_y$. These spatial frequencies are applied to the Eq. (2.75) and lead to the subsequent calculations

$$\begin{aligned}
\left(k_x^2 + k_y^2\right)^2 &= k_0^4, \\
(-\lambda_x^2 + \lambda_y^2)^2 &= k_0^4, \\
-\lambda_x^2 + \lambda_y^2 &= k_0^2, \qquad \text{due to} \qquad \lambda_x, \lambda_y \in \mathbb{R} && (2.89) \\
\lambda_x^2 &= -k_0^2 + \lambda_y^2, && (2.90) \\
\lambda_y &= \pm\sqrt{k_0^2 - k_x^2} = \pm\lambda_{x0}. && (2.91)
\end{aligned}$$

Analogously to case 1, two solutions are assumed and Eq. (2.68) can be written as

$$W_{12}(x,y) = \hat{W}e^{\pm\lambda_{x0}x}e^{j\lambda_y y}, \tag{2.92}$$

$$W(x,y) = \sum_{n=1}^{2} W_n(x,y), \tag{2.93}$$

$$W_1(x,y) = \hat{W}_1 e^{\lambda_{x0}x}e^{j\lambda_y y}, \tag{2.94}$$

$$W_2(x,y) = \hat{W}_2 e^{-\lambda_{x0}x}e^{j\lambda_y y}. \tag{2.95}$$

Using the **fifth boundary condition** $W(x,y) = 0$ for $x = 0$ leads to

$$\hat{W}_1 = -\hat{W}_2 = \frac{\hat{W}}{2}, \tag{2.96}$$

$$W(x,y) = \frac{\hat{W}}{2} e^{j\lambda_y y} \left(e^{\lambda_{x0} x} - e^{-\lambda_{x0} x}\right), \tag{2.97}$$

$$W(x,y) = \hat{W} e^{j\lambda_y y} \sinh(\lambda_{x0} x). \tag{2.98}$$

As a next step this equation can be evaluated at the boundary $x = L_x$, the **sixth boundary condition** $W(L_x, y) = 0$

$$W(x,y) = \hat{W}_1 e^{j\lambda_y y} \sinh(\lambda_{x0} L_x) = 0, \tag{2.99}$$

$$\Rightarrow \lambda_{x0} = 0. \tag{2.100}$$

Similarly to the interpretation of case 1, the configuration of complex-valued k_x and real k_y does not lead to a meaningful solution. Additionally, due to the symmetric formulation of real k_x and complex-valued k_y case 3 also does not result in a reasonable solution.

Fourth Case: Real-valued Spatial Frequencies

The last possible configuration of the spatial frequencies can be stated as real $k_x = \lambda_x$ and real $k_y = \lambda_y$ due to the Fig. 2.11. The Eq. (2.75) can be reformulated as

$$\left(k_x^2 + k_y^2\right)^2 = k_0^4,$$

$$\lambda_x^2 + \lambda_y^2 = k_0^2, \qquad \text{due to} \qquad \lambda_x, \lambda_y \in \mathbb{R} \tag{2.101}$$

$$\lambda_y^2 = k_0^2 - \lambda_x^2, \tag{2.102}$$

$$\lambda_y = \pm\sqrt{k_0^2 - \lambda_x^2} = \pm\lambda_{y0}, \tag{2.103}$$

$$\lambda_x = \pm\sqrt{k_0^2 - \lambda_y^2} = \pm\lambda_{x0}. \tag{2.104}$$

Inserting λ_y into Eq. (2.68) leads to

$$W_{12}(x,y) = \hat{W} e^{jk_x x} e^{\pm j\lambda_{y0} y} = \hat{W} e^{j\lambda_x x} e^{\pm j\lambda_{y0} y}, \tag{2.105}$$

$$W(x,y) = \sum_{n=1}^{2} W_n(x,y), \tag{2.106}$$

$$W_1(x,y) = \hat{W}_1 e^{j\lambda_x x} e^{j\lambda_{y0} y}, \tag{2.107}$$

$$W_2(x,y) = \hat{W}_2 e^{j\lambda_x x} e^{-j\lambda_{y0} y}. \tag{2.108}$$

The **first boundary condition** $W(x,y) = 0$ for $y = 0$ simplifies Eq. (2.106) to

$$\hat{W}_1 = -\hat{W}_2 = \frac{\hat{W}^{(y)}}{2}, \tag{2.109}$$

$$W(x,y) = \frac{\hat{W}^{(y)}}{2} e^{j\lambda_x x} \left(e^{j\lambda_{y0} y} - e^{-j\lambda_{y0} y} \right), \tag{2.110}$$

$$W(x,y) = \hat{W}^{(y)} e^{j\lambda_x x} j \sin(\lambda_{y0} y). \tag{2.111}$$

Additionally, the **second boundary condition** $W(x,y) = 0$ for $y = L_y$ leads to:

$$W(x,y) = \hat{W}^{(y)} e^{\lambda_x x} j \sin(\lambda_{y0} y) = 0, \tag{2.112}$$

$$\lambda_{y0} L_y = \mu_y \pi, \tag{2.113}$$

$$\lambda_{y0} = \mu_y \frac{\pi}{L_y}. \tag{2.114}$$

The Eq. (2.114) describes the spatial frequency in dependence of the wave number μ_y and the length of the plate in y-direction. Although the spatial frequency λ_{y0} is already defined, two further boundary conditions of a simply supported plate must be fullfilled. The third one is known as

$$\frac{\partial^2}{\partial y^2} W(x,y) = 0 \qquad \text{for} \qquad y = 0, \tag{2.115}$$

$$\frac{\partial^2}{\partial y^2} W(x,y) = \hat{W}^{(y)} e^{j\lambda_x x} j \sin(\lambda_{y0} y) \left(-\lambda_{y0}^2 \right), \tag{2.116}$$

$$\frac{\partial^2}{\partial y^2} W(x,0) = 0. \tag{2.117}$$

Obviously, also the **third boundary condition** is fulfilled. The fourth is similarly defined for $y = L_y$. Using the Eq. (2.114) it follows

$$\frac{\partial^2}{\partial y^2} W(x,y) = 0 \qquad \text{for} \qquad y = L_y, \tag{2.118}$$

$$\frac{\partial^2}{\partial y^2} W(x,L_y) = \hat{W}^{(y)} e^{j\lambda_x x} j \underbrace{\sin(\mu_y \pi)}_{=0} \left(-\lambda_{y0}^2\right) = 0. \tag{2.119}$$

Hence, the **fourth boundary condition** is valid as well. This means that waves in y-direction can be fully described by Eq. (2.111). As a next step the waves in x-direction must be investigated more precisely. Reusing Eq. (2.104) λ_x can be calculated as

$$\lambda_x = \pm\sqrt{k_0^2 - \lambda_y^2} = \pm\lambda_{x_0}. \tag{2.120}$$

Then the deflection of Eq. (2.111) is recalculated as

$$W_{34}(x,y) = \hat{W}_{34} e^{\pm j\lambda_{x_0} x} \hat{W}^{(y)} j \sin(\lambda_{y_0} y), \tag{2.121}$$

$$W(x,y) = \sum_{n=3}^{4} W_n(x,y), \tag{2.122}$$

$$W_3(x,y) = \hat{W}_3 e^{j\lambda_{x_0} x} \hat{W}^{(y)} j \sin(\lambda_{y_0} y), \tag{2.123}$$

$$W_4(x,y) = \hat{W}_4 e^{j\lambda_{x_0} x} \hat{W}^{(y)} j \sin(\lambda_{y_0} y), \tag{2.124}$$

$$W(x,y) = j\hat{W}^{(y)} \sin(\lambda_{y_0} y) \left[\hat{W}_3 e^{j\lambda_{x_0} x} + \hat{W}_4 e^{-j\lambda_{x_0} x}\right]. \tag{2.125}$$

By inserting the **fifth boundary condition** of $W(x,y) = 0$ for $x = 0$ it can be followed

$$\hat{W}_3 = -\hat{W}_4 = \frac{\hat{W}^{(x)}}{2}, \tag{2.126}$$

$$W(x,y) = j\hat{W}^{(y)} \sin(\lambda_{y_0} y) j\hat{W}^{(x)} \sin(\lambda_{x_0} x). \tag{2.127}$$

The spatial frequency of λ_x can be solved by applying the **sixth boundary condition** $W(x,y) = 0$ for $x = L_x$

$$\lambda_{x_0} L_x = \mu_x \pi, \tag{2.128}$$

$$\lambda_{x_0} = \mu_x \frac{\pi}{L_x}. \tag{2.129}$$

Although the spatial frequency λ_{x_0} is determined, two further boundary conditions must be tested. The **seventh boundary condition** is stated as

$$\frac{\partial^2}{\partial x^2}W(x,y) = 0 \qquad \text{for} \qquad x = 0, \tag{2.130}$$

$$\frac{\partial^2}{\partial x^2}W(x,y) = j\hat{W}^{(y)}\sin(\lambda_{y_0}y)j\hat{W}^{(x)}\sin(\lambda_{x_0}x)\left(-\lambda_{x_0}^2\right), \tag{2.131}$$

$$\frac{\partial^2}{\partial x^2}W(x,0) = 0 \tag{2.132}$$

and is thus valid. The last, **eighth boundary condition** is applied together with Eq. (2.129)

$$\frac{\partial^2}{\partial x^2}W(x,y) = 0 \qquad \text{for} \qquad x = L_x, \tag{2.133}$$

$$\frac{\partial^2}{\partial x^2}W(L_x,y) = j\hat{W}^{(y)}\sin(\lambda_{y_0}y)j\hat{W}^{(x)}\underbrace{\sin(\mu_x\pi)}_{=0}\left(-\lambda_{x_0}^2\right) = 0. \tag{2.134}$$

As a result Eq. (2.127) is valid for all eight boundary conditions and can be stated as

$$W(x,y) = -\hat{W}^{(y)}\sin(\lambda_{y_0}y)\hat{W}^{(x)}\sin(\lambda_{x_0}x), \tag{2.135}$$

$$W(x,y) = \hat{A}\sin(\lambda_{y_0}y)\sin(\lambda_{x_0}x) \qquad \text{with} \qquad \hat{A} = -\hat{W}^{(x)}\hat{W}^{(y)}, \tag{2.136}$$

$$k_0^2 = \lambda_{x_0}^2 + \lambda_{y_0}^2 = \left(\mu_x\frac{\pi}{L_x}\right)^2 + \left(\mu_y\frac{\pi}{L_y}\right)^2. \tag{2.137}$$

2.2.2 Proof of the Dynamic Solution

In order to doublecheck the developed solution for $W(x,y)$ in Sec. 2.2.1 the general form of the PDE is inserted into Eq. (2.67)

$$\Delta\Delta W(x,y) - k_0^4 W(x,y) = 0, \tag{2.138}$$

whereas $W(x,y)$ is known from Eq. (2.136) and (2.137). Further the Laplace operator can be calculated with regard to Eq. (2.70)

$$\Delta\Delta W(x,y) = \frac{\partial^4}{\partial x^4}W(x,y) + \frac{\partial^4}{\partial x^2\partial y^2}W(x,y) + \frac{\partial^4}{\partial y^4}W(x,y). \tag{2.139}$$

and the partial derivatives can be obtained from (2.127)

$$\frac{\partial^4}{\partial x^4}W(x,y) = \lambda_{x0}^4 \hat{A}\sin(\lambda_{y0}y)\sin(\lambda_{x0}x), \tag{2.140}$$

$$\frac{\partial^4}{\partial x^2 \partial y^2}W(x,y) = \lambda_{x0}^2\lambda_{y0}^2 \hat{A}\sin(\lambda_{y0}y)\sin(\lambda_{x0}x), \tag{2.141}$$

$$\frac{\partial^4}{\partial y^4}W(x,y) = \lambda_{y0}^4 \hat{A}\sin(\lambda_{y0}y)\sin(\lambda_{x0}x). \tag{2.142}$$

By inserting these equations in Eq. (2.139) it can be shown

$$\begin{aligned}\Delta\Delta W(x,y) &= \left(\lambda_{x0}^4 + 2\lambda_{x0}^2\lambda_{y0}^2 + \lambda_{y0}^4\right) W(x,y) \\ &= \left(\lambda_{x0}^2 + \lambda_{y0}^2\right)^2 W(x,y).\end{aligned} \tag{2.143}$$

With the knowledge of Eq. (2.75) it can be followed

$$k_0^2 = \lambda_{x0}^2 + \lambda_{y0}^2, \tag{2.144}$$

$$\Delta\Delta W(x,y) = k_0^4 W(x,y), \tag{2.145}$$

$$\Rightarrow \Delta\Delta W(x,y) - k_0^4 W(x,y) = 0. \tag{2.146}$$

Thus, Eq. (2.138) is valid.

2.2.3 Navier's Static Solution for Simply Supported Plates

As a further reference Navier's solution shows a way to solve the PDE for rectangular plates for the static case [10, 8, p.297][8, p.141-143]. For the bending problem the Fourier Series for the load and deflection is introduced

$$\text{load:} \qquad p(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{mn} \sin\left(\frac{m\pi x}{L_x}\right)\sin\left(\frac{n\pi y}{L_y}\right), \tag{2.147}$$

$$\text{deflection:} \qquad w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{m\pi x}{L_x}\right)\sin\left(\frac{n\pi y}{L_y}\right). \tag{2.148}$$

The coefficients p_{mn} and a_{mn} have to be determined. Here similarly to Sec. 2.2.1, the deflection and the coefficients of a_{mn} must fulfill the PDE (2.41) and the boundary conditions of simply supported plates (2.57) and (2.58). In order to obtain the Fourier coefficients p_{mn} the Eq. (2.147) is multiplied by

$$\sin\left(\frac{m'\pi x}{L_x}\right)\sin\left(\frac{n'\pi y}{L_y}\right)\mathrm{d}x\mathrm{d}y. \tag{2.149}$$

By integration from 0 to L_x and 0 to L_y the equation leads to

$$\int_0^{L_y}\int_0^{L_x} p(x,y)\sin\left(\frac{m'\pi x}{L_x}\right)\sin\left(\frac{n'\pi y}{L_y}\right)\mathrm{d}x\mathrm{d}y = \tag{2.150}$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{mn}\sin\left(\frac{m\pi x}{L_x}\right)\sin\left(\frac{n\pi y}{L_y}\right)\sin\left(\frac{m'\pi x}{L_x}\right)\sin\left(\frac{n'\pi y}{L_y}\right)\mathrm{d}x\mathrm{d}y. \tag{2.151}$$

Direct integration in x-direction shows that

$$\int_0^{L_x}\sin\left(\frac{m\pi x}{L_x}\right)\sin\left(\frac{m'\pi x}{L_x}\right) = \frac{L_x}{2} \qquad (m = m'), \tag{2.152}$$

$$\int_0^{L_x}\sin\left(\frac{m\pi x}{L_x}\right)\sin\left(\frac{m'\pi x}{L_x}\right) = 0 \qquad (m \neq m'), \tag{2.153}$$

and in y-direction

$$\int_0^{L_y}\sin\left(\frac{n\pi y}{L_y}\right)\sin\left(\frac{n'\pi y}{L_y}\right) = \frac{L_y}{2} \qquad (m = m'), \tag{2.154}$$

$$\int_0^{L_y}\sin\left(\frac{n\pi y}{L_y}\right)\sin\left(\frac{n'\pi y}{L_y}\right) = 0 \qquad (m \neq m'). \tag{2.155}$$

Therefore, the coefficients of the Fourier expansion follow as

$$p_{mn} = \frac{4}{L_x L_y}\int_0^{L_y}\int_0^{L_x} p(x,y)\sin\left(\frac{m\pi x}{L_x}\right)\sin\left(\frac{n\pi y}{L_y}\right)\mathrm{d}x\mathrm{d}y. \tag{2.156}$$

For the determination of the coefficients a_{mn} the PDE (2.41) is reused

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{a_{mn}\left[\left(\frac{m\pi}{L_x}\right)^4 + 2\left(\frac{m\pi}{L_x}\right)^2\left(\frac{n\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_y}\right)^4\right] - \frac{p_m n}{D}\right\} \tag{2.157}$$

$$\sin\left(\frac{m\pi x}{L_x}\right)\sin\left(\frac{n\pi y}{L_y}\right) = 0. \tag{2.158}$$

If this equation shall hold for all x and y, the following equation must be valid

$$a_{mn}\pi^4\left(\frac{m^2}{L_x^2} + \frac{n^2}{L_y^2}\right)^2 - \frac{p_{mn}}{D} = 0. \tag{2.159}$$

Solving Eq. (2.159) a_{mn} leads to the coefficients

$$a_{mn} = \frac{1}{\pi^4 D} \frac{p_{mn}}{\left[\frac{m^2}{L_x^2} + \frac{n^2}{L_y^2}\right]^2}. \tag{2.160}$$

Now the deflection can be represented as

$$w(x,y) = \frac{1}{\pi^4 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_{mn}}{\left[\frac{m^2}{L_x^2} + \frac{n^2}{L_y^2}\right]^2} \sin\left(\frac{m\pi x}{L_x}\right) \sin\left(\frac{n\pi y}{L_y}\right) \tag{2.161}$$

in terms of a double superposition of sin-functions that depend on the lengths L_x and L_y and the coefficients p_{mn}. The Eq. (2.161) represents a solution for simply supported rectangular plates that are bended under a certain loading. Notify that this solution is developed under a static consideration.

Chapter 2 has shown the physical behaviour of plates and analytical solutions for the dynamic and static case for simple boundary conditions were developed. In the next section an additional method for solving certain time-dependent PDEs is introduced.

Chapter 3

Functional Transformation Method (FTM)

Originally, the FTM for multi-dimensional systems have been developed for musical applications and implementations [1]. Moreover, the method and its benefits can also be applied to general physical systems that can be described in form of the PDE (see Eq. (3.1)). This Sec. 3 explains the general procedure of the Functional Transformation Method for the n-dimensional application. The transfer function model shows a method of converting time and space dynamics into a function of a complex frequency variables. Similarly to the Laplace-Transformation of time variables into corresponding frequencies variables, the Sturm-Louiville-Transformation turns the spatial dynamic problem into a function of a corresponding spatial frequency variable. Via this method an algebraic solution is obtained that can be developed further to a suitable synthesis algorithm for a digital implementation.

3.1 Physical Description for Multi-dimensional Systems

The following section provides a procedure that refers to PDEs depending on time and space that have the general vectorial form of

$$[\boldsymbol{C}\frac{\partial}{\partial t} - L]\boldsymbol{y}(\boldsymbol{x}, t) = \boldsymbol{f}_e(\boldsymbol{x}, t), \tag{3.1}$$

whereas $\boldsymbol{x} \in V$. The differential operator L is defined as

$$L = \boldsymbol{A} + \nabla\boldsymbol{B}. \tag{3.2}$$

The matrix $\boldsymbol{C}$ contains all physical parameters regarding the time derivatives and $\boldsymbol{B}$ all physical parameters regarding the space derivatives respectively. Further physical parameters are stored in the matrix $\boldsymbol{A}$. The vector $\boldsymbol{y}(\boldsymbol{x}, t)$ describes the physical quantities of the system depending on space and time. The vector $\boldsymbol{f}_e(\boldsymbol{x}, t)$ is the excitation function.

3.2 Steps of the FTM

In Fig. 2.11 the sequence of the single steps of the transfer function method are depicted. The genereal idea is to transform the given PDE into an algebraic equation by the Laplace and Sturm-Liouville Transformation. Afterwards, this equation is solved for the output signal that leads to the multidimensional transfer function. By the Impulse-Invariant Transformation that function is discretized. The synthesis algorithm is then obtained by the inverse Sturm-Liouville and Laplace Transformation. These steps are explained in the next subsections in detail.

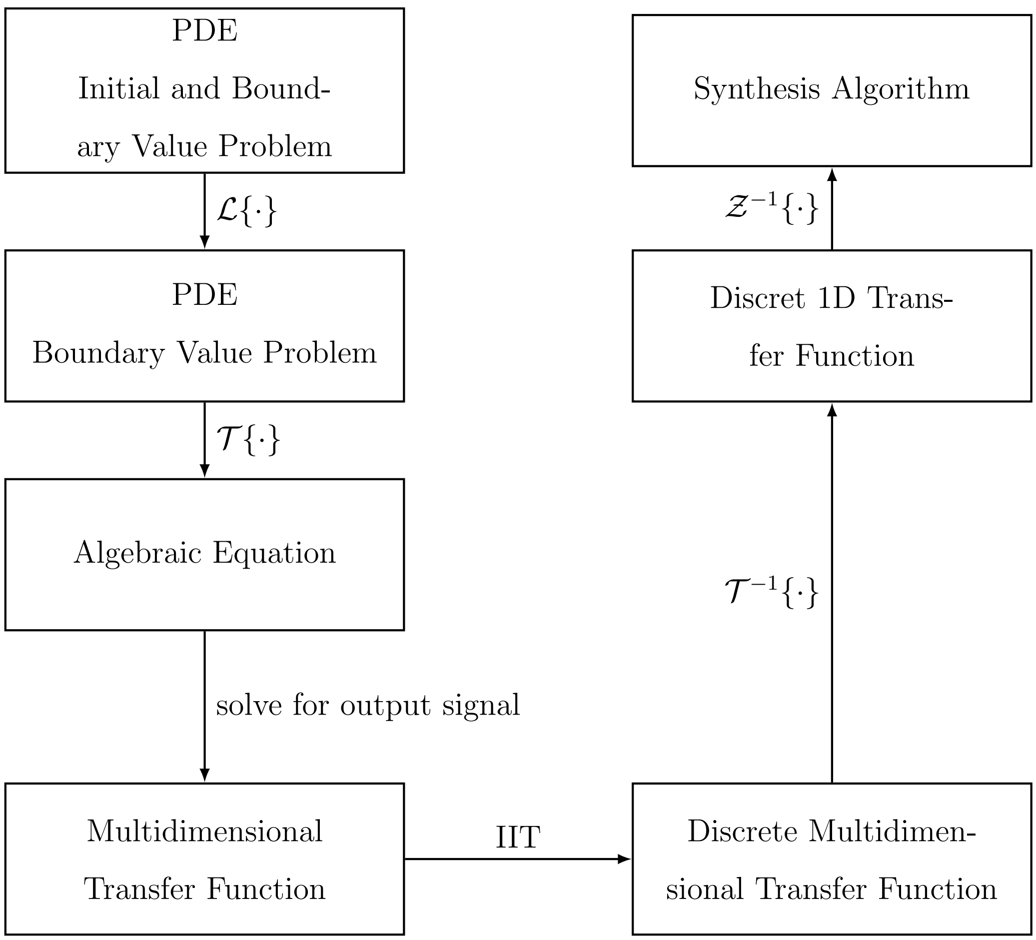


Figure 3.1: Single Steps of the FTM. $\mathcal{L}\{\cdot\}$: Laplace Transformation w.r.t initial conditions; $\mathcal{T}\{\cdot\}$: Sturm-Liouville Transformation w.r.t. boundary conditions; IIT: Impulse-Invariant Transformation [14, S. 37].

3.3 Initial and Boundary Conditions

In order to solve partial differential equations intial and boundary conditions are generally required. The initial conditions describe the state of the system at the time $t = 0$ and can be defined on the whole spatial area $\boldsymbol{x} \in V$

$$\boldsymbol{y}(\boldsymbol{x}, 0) = \boldsymbol{y}_i(\boldsymbol{x}). \tag{3.3}$$

For the description of the relevant boundary conditions a new matrix $\boldsymbol{f}_{\mathrm{b}}^{\mathrm{H}}$ is introduced

$$\boldsymbol{f}_{\mathrm{b}}^{\mathrm{H}}\boldsymbol{y}(\boldsymbol{x}, t) = \boldsymbol{\phi}(\boldsymbol{x}, t), \qquad \boldsymbol{x} \in \partial\boldsymbol{V}. \tag{3.4}$$

Here the superscript denotes the property of an hermitian matrix. For a given boundary operator $\boldsymbol{f}_{\mathrm{b}}^{\mathrm{H}}$ and vector $\boldsymbol{y}$ the boundary excitation at the boundary is represented by the vector $\boldsymbol{\phi}(\boldsymbol{x}, t)$.

3.4 Laplace Transformation

The first transformation with regard to Fig. 2.11 is a one-sided Laplace transformation using the initial conditions

$$Y(s) = \mathcal{L}\left\{y(t)\right\} = \int_0^\infty y(t)e^{-st}\mathrm{d}t. \tag{3.5}$$

The variable s denotes the temporal frequency variable. On the basis of Eq. (3.5) the differentiation theorem is defined as

$$\mathcal{L}\left\{\frac{\partial}{\partial t}y(t)\right\} = s\mathcal{L}\left\{y(t)\right\} - y(0), \tag{3.6}$$

$$\mathcal{L}\left\{\frac{\partial^2}{\partial t^2}y(t)\right\} = s^2\mathcal{L}\left\{y(t)\right\} - sy(0) - \frac{\partial}{\partial t}y(t)\Big|_{t=0}. \tag{3.7}$$

The Laplace Transform of Eq. (3.5) is applied to the vector PDE (3.1) and to the boundary conditions of Eq. (3.4), which leads to

$$[s\boldsymbol{C} - L]\boldsymbol{Y}(x, s) = \boldsymbol{F}_e(x, s) + \boldsymbol{C}\boldsymbol{y}_{\mathrm{i}}(\boldsymbol{x}) \qquad \boldsymbol{x} \in \boldsymbol{V}, \tag{3.8}$$

$$\boldsymbol{F}_{\mathrm{b}}^{\mathrm{H}}\boldsymbol{Y}(\boldsymbol{x}, s) = \boldsymbol{\Phi}(\boldsymbol{x}, s) \qquad \boldsymbol{x} \in \partial\boldsymbol{V}. \tag{3.9}$$

3.5 Sturm-Liouville Transformation

After having transformed the PDE with initial and boundary conditions to a PDE with only a boundary value problem an appropriate transformation has to be developed in order to obtain an algebraic solution. This algebaric equation does not depend afterwards on a boundary value problem either. The Sturm-Liouville Transformation (SLT) is a linear integral transform that deals the space derivatives similarly to the time derivatives of the Laplace transform, whereas the transformation kernel is not known yet and must be determined dependend on the PDE and the given boundary conditions.

3.5.1 Definition of the Sturm-Liouville Transformation

The SLT can be defined in terms of a scalar product and integral form

$$\mathcal{T}\{\boldsymbol{Y}(\boldsymbol{x},s)\} = \bar{Y}(\mu,s) = \langle \boldsymbol{C}\boldsymbol{Y}(\boldsymbol{x},s), \tilde{\boldsymbol{K}}(\boldsymbol{x},\mu)\rangle = \int_V \tilde{\boldsymbol{K}}^H(\boldsymbol{x},\mu)\boldsymbol{C}\boldsymbol{Y}(\boldsymbol{x},s)\mathrm{d}x. \quad (3.10)$$

Here, the adjoint kernel function is called $\tilde{\boldsymbol{K}}$ and represents the eigenfunction of the transformation. In general, the kernel is unknown and has to be determined regarding a specific eigenvalue problem. The integer index μ is the index of a discrete spatial frequency variable s_μ. There is also an primal transformation kernel $\boldsymbol{K}$ (see Sec. 3.7) for the inverse transformation.

3.5.2 Application of the Sturm-Liouville Transformation

Applying the SLT to the Eq. (3.8) leads to

$$s\bar{Y}(\mu,s) - \langle L\boldsymbol{Y}(\boldsymbol{x},s), \tilde{\boldsymbol{K}}(\boldsymbol{x},\mu)\rangle = \bar{F}_e(\mu,s) + \bar{y}_i(\mu), \quad (3.11)$$

with the transformed terms

$$\bar{Y}(\mu, s) = \langle \boldsymbol{C}\boldsymbol{Y}(\boldsymbol{x}, s), \tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu)\rangle, \tag{3.12}$$

$$\bar{F}_e(\mu, s) = \langle \boldsymbol{F_e}(\boldsymbol{x}, s), \tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu)\rangle, \tag{3.13}$$

$$\bar{y}_i(\mu) = \langle \boldsymbol{C}\boldsymbol{y_i}(\boldsymbol{x}), \tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu)\rangle. \tag{3.14}$$

The second term of Eq. (3.11) is unknown so far, because no differentiation theorem for the SLT is stated. This theorem is developed analogously to the differentiation theorem of the Laplace transform in Eq. (3.6) in the next sections.

3.5.3 Introduction of a Second Differential Operator

In order to derive a differentiation theorem a second differential operator is introduced. The second term of Eq. (3.11) can be rewritten with $L = \boldsymbol{A} + \nabla \boldsymbol{B}$ from Eq. (3.2)

$$\langle L\boldsymbol{Y}(\boldsymbol{x}, s), \tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu)\rangle = \langle \boldsymbol{A}\boldsymbol{Y}(\boldsymbol{x}, s), \tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu)\rangle + \langle \boldsymbol{B}(\nabla \boldsymbol{Y})(\boldsymbol{x}, s), \tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu)\rangle. \tag{3.15}$$

Investigating this equation shows that the second term on the right side can be solved by intergration of parts

$$\begin{aligned} \langle \boldsymbol{B}(\nabla \boldsymbol{Y})(\boldsymbol{x}, s), \tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu)\rangle &= \int_V \tilde{\boldsymbol{K}}^H(\boldsymbol{x}, \mu)\boldsymbol{B}(\nabla \boldsymbol{Y}(\boldsymbol{x}, s))\mathrm{d}x \\ &= \int_{\partial V} \tilde{\boldsymbol{K}}^H(\boldsymbol{x}, \mu)\boldsymbol{B}(\boldsymbol{Y}(\boldsymbol{x}, s))\mathrm{d}x - \int_V \nabla\tilde{\boldsymbol{K}}^H(\boldsymbol{x}, \mu)\boldsymbol{B}\boldsymbol{Y}(\boldsymbol{x}, s)\mathrm{d}x \\ &= \int_{\partial V} \tilde{\boldsymbol{K}}^H(\boldsymbol{x}, \mu)\boldsymbol{B}(\boldsymbol{Y}(\boldsymbol{x}, s))\mathrm{d}x - \langle \boldsymbol{B}\boldsymbol{Y}(\boldsymbol{x}, s), \nabla\tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu)\rangle. \end{aligned} \tag{3.16}$$

This result can be used to rearrange Eq. (3.15) and leads to

$$\begin{aligned} \langle L\boldsymbol{Y}(\boldsymbol{x}, s), \tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu)\rangle =& \langle \boldsymbol{A}\boldsymbol{Y}(\boldsymbol{x}, s), \tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu)\rangle - \langle \boldsymbol{B}\boldsymbol{Y}(\boldsymbol{x}, s), \nabla\tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu)\rangle \\ &+ \int_{\partial V} \tilde{\boldsymbol{K}}^H(\boldsymbol{x}, \mu)\boldsymbol{B}(\boldsymbol{Y}(\boldsymbol{x}, s))\mathrm{d}x \\ =& \langle \boldsymbol{Y}(\boldsymbol{x}, s), \boldsymbol{A}^H\tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu)\rangle - \langle \boldsymbol{Y}(\boldsymbol{x}, s), \boldsymbol{B}^H\nabla\tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu)\rangle \\ &+ \int_{\partial V} \tilde{\boldsymbol{K}}^H(\boldsymbol{x}, \mu)\boldsymbol{B}(\boldsymbol{Y}(\boldsymbol{x}, s))\mathrm{d}x. \end{aligned} \tag{3.17}$$

Introducing a new operator $\tilde{L} = \boldsymbol{A}^H - \boldsymbol{B}^H\nabla$ the equation can be reformulated to

$$\langle L\boldsymbol{Y}(\boldsymbol{x},s), \tilde{\boldsymbol{K}}(\boldsymbol{x},\mu)\rangle = \langle \boldsymbol{Y}(\boldsymbol{x},s), \tilde{L}\tilde{\boldsymbol{K}}(\boldsymbol{x},\mu)\rangle + \int_{\partial V} \tilde{\boldsymbol{K}}^H(\boldsymbol{x},\mu)\boldsymbol{B}\boldsymbol{Y}(\boldsymbol{x},s)\mathrm{d}x. \quad (3.18)$$

3.5.4 Primal and Adjoint Operator

In this section the properties of this operator $\tilde{L}$ are considered in more detail. Substituting $\boldsymbol{Y}(\boldsymbol{x},s)$ by $\boldsymbol{K}(\boldsymbol{x},\mu)$ results in

$$\langle L\boldsymbol{K}(\boldsymbol{x},\mu), \tilde{\boldsymbol{K}}(\boldsymbol{x},\mu)\rangle - \langle \boldsymbol{K}(\boldsymbol{x},\mu), \tilde{L}\tilde{\boldsymbol{K}}(\boldsymbol{x},\mu)\rangle = \int_{\partial V} \tilde{\boldsymbol{K}}^H(\boldsymbol{x},\mu)\boldsymbol{B}\boldsymbol{K}(\boldsymbol{x},\mu)\mathrm{d}x. \quad (3.19)$$

Now it can be stated that the differential operators L and $\tilde{L}$ fullfill the requirements for adjoint operators if the right integral equals zero

$$\int_{\partial V} \tilde{\boldsymbol{K}}^H(\boldsymbol{x},\mu)\boldsymbol{B}\boldsymbol{K}(\boldsymbol{x},\mu)\mathrm{d}x = 0. \quad (3.20)$$

Hereby the operator L is called primal operator and $\tilde{L}$ is defined as the adjoint operator. Similarly, the kernel $\boldsymbol{K}(\boldsymbol{x},s)$ is the primal kernel and $\tilde{\boldsymbol{K}}(\boldsymbol{x},s)$ the adjoint kernel.

3.5.5 Boundary Conditions for the Kernel Functions

In order to proof the condition of Eq. (3.20) a closer look of the boundary condition is needed. To this end, new integrals are introduced that contain the following matrix combination: $\boldsymbol{W}\boldsymbol{F_b}^H$. The Eq. (3.20) can be reformulated as follows (dependencies of the vectors are omitted here for brevity)

$$\begin{aligned}\int_{\partial V} \tilde{\boldsymbol{K}}^H\boldsymbol{B}\boldsymbol{K}\mathrm{d}x &= \int_{\partial V} \tilde{\boldsymbol{K}}^H\boldsymbol{W}\boldsymbol{F_b}^H\boldsymbol{K}\mathrm{d}x + \int_{\partial V} \tilde{\boldsymbol{K}}^H\boldsymbol{B}\boldsymbol{K}\mathrm{d}x - \int_{\partial V} \tilde{\boldsymbol{K}}^H\boldsymbol{W}\boldsymbol{F_b}^H\boldsymbol{K}\mathrm{d}x \\ &= \int_{\partial V} \tilde{\boldsymbol{K}}^H\boldsymbol{W}\boldsymbol{F_b}^H\boldsymbol{K}\mathrm{d}x + \int_{\partial V} \tilde{\boldsymbol{K}}^H(\boldsymbol{B} - \boldsymbol{W}\boldsymbol{F_b}^H)\boldsymbol{K}\mathrm{d}x \\ &= \int_{\partial V} \tilde{\boldsymbol{K}}^H\boldsymbol{W}\boldsymbol{F_b}^H\boldsymbol{K}\mathrm{d}x + \int_{\partial V} \left[(\boldsymbol{B} - \boldsymbol{W}\boldsymbol{F_b}^H)^H\tilde{\boldsymbol{K}}\right]^H\boldsymbol{K}\mathrm{d}x.\end{aligned} \quad (3.21)$$

For a generalization of the results an arbitrary matrix $\boldsymbol{W}$ is used. Futhermore, a matrix of boundary conditions for the adjoint kernel is defined as

$$\tilde{\boldsymbol{F}}_{\boldsymbol{b}} = \boldsymbol{B} - \boldsymbol{W}\boldsymbol{F}_{\boldsymbol{b}}{}^{H} \tag{3.22}$$

and can be used to reformulate Eq. (3.21) to

$$\int_{\partial V} \tilde{\boldsymbol{K}}^{\mathrm{H}} \boldsymbol{B}\boldsymbol{K} \mathrm{d}x = \int_{\partial V} \tilde{\boldsymbol{K}}^{\mathrm{H}} \boldsymbol{W} \left[\boldsymbol{F}_{\boldsymbol{b}}{}^{\mathrm{H}} \boldsymbol{K}\right] \mathrm{d}x + \int_{\partial V} \left[\tilde{\boldsymbol{F}}_{\boldsymbol{b}}{}^{\mathrm{H}} \tilde{\boldsymbol{K}}\right]^{H} \boldsymbol{K} \mathrm{d}x. \tag{3.23}$$

Now it can be seen that the condition of adjoint operators is valid if the boundary conditions for both kernels are fulfilled

$$\boldsymbol{F}_{\boldsymbol{b}}{}^{\mathrm{H}} \boldsymbol{K} = 0, \qquad \boldsymbol{x} \in \partial V, \tag{3.24}$$

$$\tilde{\boldsymbol{F}}_{\boldsymbol{b}}{}^{\mathrm{H}} \tilde{\boldsymbol{K}} = 0, \qquad \boldsymbol{x} \in \partial V. \tag{3.25}$$

3.5.6 Eigenvalue Problem

The Eq. (3.18) can be now considered in detail. The first term on the right hand side shall be solved by a differentiation theorem similarly to the Laplace transformation. Therefore, an eigenvalue problem for the adjoint kernel can be stated as

$$\tilde{L}\tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu) = s_\mu^* \boldsymbol{C}^H \tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu), \qquad \boldsymbol{x} \in V. \tag{3.26}$$

Analogously, the eigenvalue problem for the primal kernel $\boldsymbol{K}(x, \mu)$ can be established in the form

$$L\boldsymbol{K}(\boldsymbol{x}, \mu) = s_\mu \boldsymbol{C}\boldsymbol{K}(\boldsymbol{x}, \mu), \qquad \boldsymbol{x} \in V. \tag{3.27}$$

3.5.7 Differentiation Theorem

After having established a proper eigenvalue problem the first term on the right hand side of Eq. (3.18) can be solved

$$\langle L\boldsymbol{Y}(\boldsymbol{x}, s), \tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu)\rangle = \langle \boldsymbol{Y}(\boldsymbol{x}, s), s_\mu^* \boldsymbol{C}^H \tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu)\rangle \tag{3.28}$$

$$= s_\mu \langle \boldsymbol{C}\boldsymbol{Y}(\boldsymbol{x}, s), \tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu)\rangle = s_\mu \bar{Y}(\mu, s). \tag{3.29}$$

This differentiation theorem leads again to a scalar product and a multiplication by the eigenvalue s_μ.

3.5.8 Boundary Term

Revisiting Eq. (3.18) the second term on the right hand side can be further solved and reformulated with the knowledge of Eq. (3.24)

$$\int_{\partial V} \tilde{\boldsymbol{K}}^H \boldsymbol{B}\boldsymbol{Y} \mathrm{d}x = \int_{\partial V} \left[\tilde{\boldsymbol{F}}_{\mathrm{b}} \tilde{\boldsymbol{K}}\right]^H \boldsymbol{Y} \mathrm{d}x + \int_{\partial V} \tilde{\boldsymbol{K}}^H \boldsymbol{W} \left[\boldsymbol{F}_{\mathrm{b}}{}^H \boldsymbol{Y}\right] \mathrm{d}x. \tag{3.30}$$

Exploiting the kernel boundary condition from Eq. (3.25) the first term on the right hand side vanishes. Moreover, the second term can be rewritten in terms of the boundary conditions of Sec. 3.3 as

$$\int_{\partial V} \tilde{\boldsymbol{K}}^H \boldsymbol{W} \left[\boldsymbol{F}_{\mathrm{b}}{}^H \boldsymbol{Y}\right] \mathrm{d}x = \int_{\partial V} \tilde{\boldsymbol{K}}^H(\boldsymbol{x}, \mu) \boldsymbol{W} \boldsymbol{\Phi}(x, s) \mathrm{d}x = \bar{\Phi}(\mu, s). \tag{3.31}$$

3.5.9 Transform Domain Representation

Based on both transforms – the Laplace and Sturm-Liouville – Eq. (3.11) can be transformed and formulated with the multiplications of the complex frequency variable s and the eigenvalues s_μ

$$s\bar{Y}(\mu, s) - s_\mu \bar{Y}(\mu, s) = \bar{F}_e(\mu, s) + \bar{y}_i(\mu) + \bar{\Phi}(\mu, s). \tag{3.32}$$

This Eq. (3.32) represents an algebraic equation and is supposed to be solved for obtaining a multidimensional transfer function.

3.6 Transfer Function Model and Discretization

Solving Eq. (3.32) for the transformed vector of variables leads to

$$Y(\mu, s) = \frac{1}{s - s_\mu} \left[\bar{F}_e(\mu, s) + \bar{y}_i(\mu) + \bar{\Phi}(\mu, s)\right], \tag{3.33}$$

with the multidimensional transfer function

$$\bar{H}(\mu, s) = \frac{1}{s - s_\mu}. \tag{3.34}$$

In order to obtain a discrete representation Eq. (3.33) can be transformed by the impulse invariant transformation (IIT). This transformation defines discrete poles in the form of

$$z_\mu = \exp(s_\mu T), \tag{3.35}$$

where T is the sampling frequency. The application of the IIT on Eq. (3.33) results in

$$Y(\mu, z) = \frac{z}{z - z_\mu} \left[\bar{F}_e(\mu, z) + \bar{y}_i(\mu) + \bar{\Phi}(\mu, z) \right], \tag{3.36}$$

with the discrete multidimensional transfer function

$$\bar{H}(\mu, z) = \frac{z}{z - z_\mu}. \tag{3.37}$$

$Y(\mu, z)$, $\bar{F}_e(\mu, z)$ and $\bar{\Phi}(\mu, z)$ can be seen as the z-domain equivalents of the transfer function, the excitation function and the boundary terms.

3.7 Inverse Sturm-Liouville Transformation and Discrete Synthesis Algorithm

Before introducing the inverse Sturm-Liouville Transformation the biorthogonality between both kernel functions must be investigated in more detail. Here, it is necessary to mention that the eigenfrequencies of both kernel functions are not equal and are defined on finite space regions

$$s_\mu, s_\nu^* \in \mathbb{Z}. \tag{3.38}$$

Under that assumption the kernel function K_μ and $\tilde{K}_\nu$ are biorthogonal w.r.t. the weighting matrix $\boldsymbol{C}$ [5]. This consideration introduces a scaling factor N_μ that is the result of the following consideration

$$\langle \boldsymbol{C}\boldsymbol{K}_\mu, \tilde{\boldsymbol{K}}_\nu \rangle = \begin{cases} N_\mu & \mu = \nu, \\ 0 & \mu \neq \nu. \end{cases} \tag{3.39}$$

Alternatively, the scaling factor is given as

$$N_\mu = \langle \boldsymbol{C}\boldsymbol{K}(\boldsymbol{x}, \mu), \tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu) \rangle. \tag{3.40}$$

Due to the biorthogonality the inverse SLT can be interpreted as a superposition over all possible eigenfrequencies

$$\mathcal{T}^{-1}\left\{\bar{Y}(\mu, s)\right\} = \boldsymbol{Y}(\boldsymbol{x}, s) = \sum_\mu \frac{1}{N_\mu} \bar{Y}(\mu, s) \boldsymbol{K}(\boldsymbol{x}, \mu). \tag{3.41}$$

The subsequent Fig. 3.2 shows a graphical interpretation of Eq. (3.41) in the continuous frequency domain.

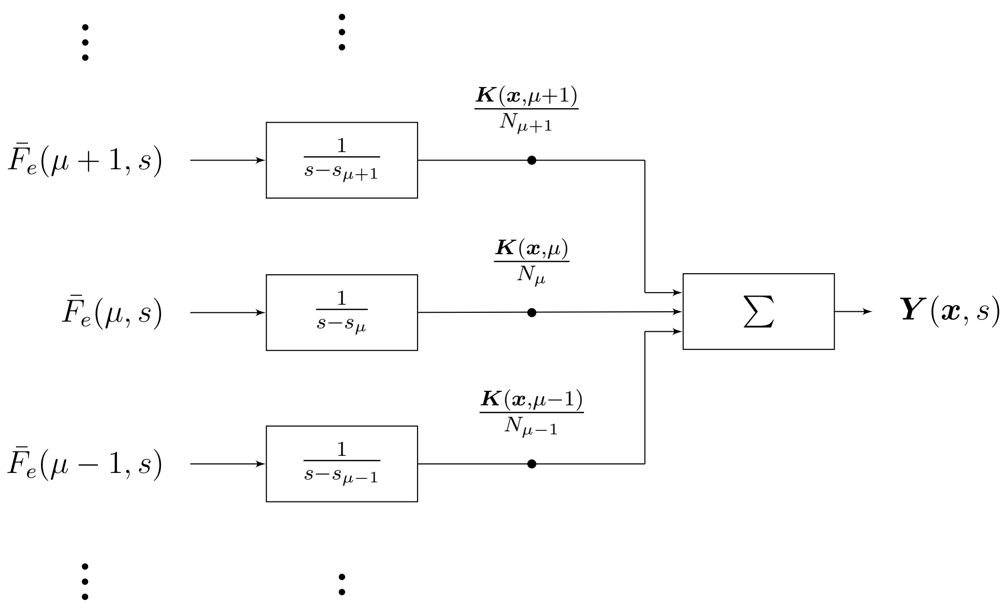


Figure 3.2: The superposition of μ first order blocks that describes the inverse SLT of Eq. (3.41). The initial and boundary terms are set to $\bar{y}_i(\mu) = 0$ and $\bar{\Phi}(\mu, z) = 0$.

The last step of Fig. 3.1 is to derive a solution in a discrete-time domain for potential computer implementation. Applying the IIT of Sec. (3.6) shows the following discrete superposition of first-order systems

$$\boldsymbol{Y}(\boldsymbol{x},z) = \sum_{\mu} \frac{1}{N_\mu} \bar{Y}(\mu,z) \boldsymbol{K}(\boldsymbol{x},\mu). \tag{3.42}$$

Remarkable is that the number of eigenfrequencies directly refers to the range of summation of the synthesis. The implementation can be done by a superposition of μ first order blocks.

3.8 Kernels and Eigenfrequencies

The transformation of Eq. (3.41) requires an explicit knowledge about the kernel functions and the spatial eigenfrequencies s_μ.

3.8.1 Derivation of Kernel Functions

As a first step the eigenvalue problem of Eq. (3.27) is revisited

$$L\boldsymbol{K}(\boldsymbol{x},\mu) = s_\mu \boldsymbol{C}^H \boldsymbol{K}(\boldsymbol{x},\mu), \qquad \boldsymbol{x} \in V, \tag{3.43}$$

where $L = \boldsymbol{A} + \nabla \boldsymbol{B}$. Consequently, this leads to

$$\nabla \boldsymbol{K}(\boldsymbol{x},\mu) = \underbrace{\boldsymbol{B}^{-1}(s_\mu \boldsymbol{C} - \boldsymbol{A})}_{=\boldsymbol{Q}} \boldsymbol{K}(\boldsymbol{x},\mu), \tag{3.44}$$

$$\nabla \boldsymbol{K}(\boldsymbol{x},\mu) = \boldsymbol{Q}\boldsymbol{K}(\boldsymbol{x},\mu), \tag{3.45}$$

where $\boldsymbol{Q}$ is defined as

$$\boldsymbol{Q} = \boldsymbol{B}^{-1}(s_\mu \boldsymbol{C} - \boldsymbol{A}). \tag{3.46}$$

For further calculations the matrix $\boldsymbol{Q}$ is introduced and is used to show the solution of the primal eigenvalue problem by means of a matrix exponential (only for one-dimensional problems)

$$\boldsymbol{K}(\boldsymbol{x},\mu) = e^{\boldsymbol{Q}\boldsymbol{x}} \boldsymbol{K}(0,\mu). \tag{3.47}$$

The vector $\tilde{\boldsymbol{K}}(0,\mu)$ is the boundary vector for the primal kernel. A proof of the application of the matrix exponential is given in the appendix B.1. Considering the adjoint eigenvalue problem and kernel the following solution can be developed based on Eq. (3.26)

$$\tilde{\boldsymbol{K}}(\boldsymbol{x},\mu) = (\boldsymbol{B}e^{\boldsymbol{Q}\boldsymbol{x}}\boldsymbol{B}^{-1})^{-H}\tilde{\boldsymbol{K}}(0,\mu), \tag{3.48}$$

whereas here $\tilde{\boldsymbol{K}}(0,\mu)$ represents the boundary vector for the adjoint kernel. Reusing the Eq. (3.24) shows also that

$$\tilde{\boldsymbol{K}}(0,\mu)^H \boldsymbol{B}\boldsymbol{K}(0,\mu) = 0. \tag{3.49}$$

Based on the Eqs. (3.48) and (3.49) there is no second matrix exponential for the calculations of the adjoint kernel $\tilde{\boldsymbol{K}}(\boldsymbol{x},\mu)$ needed.

3.8.2 Derivation of Eigenfrequencies

In order to obtain the kernel functions the calculation of the eigenvalues of the matrix $\boldsymbol{Q}$ in Eq. (3.44) is necessary. In Fig. 3.3 a possible procedure is shown. The first step is the evaluation of the boundary condition of the primal kernel at $x = L_x$. From this a rule for the eigenvalues λ can be derived. For the final calculation of the eigenfrequencies s_μ the chacteristic polynomial of the matrix $\boldsymbol{Q}$ has to be developed. Furthermore, the eigenvalues λ can be inserted in this polynomial and solved for the eigenfrequencies s_μ. The resulting equation is known as the dispersion relation that allows the determination of the eigenfrequencies.

3.8.3 Matrix Exponential

In this section the calculation of the matrix exponential $e^{\boldsymbol{Q}\boldsymbol{x}}$ is shown in the one-dimensional case meaning that the variable $\boldsymbol{x} \in [0, L_x]$. Further information for the calculation of the matrix exponential is given in [15][16]. Here the matrix exponential

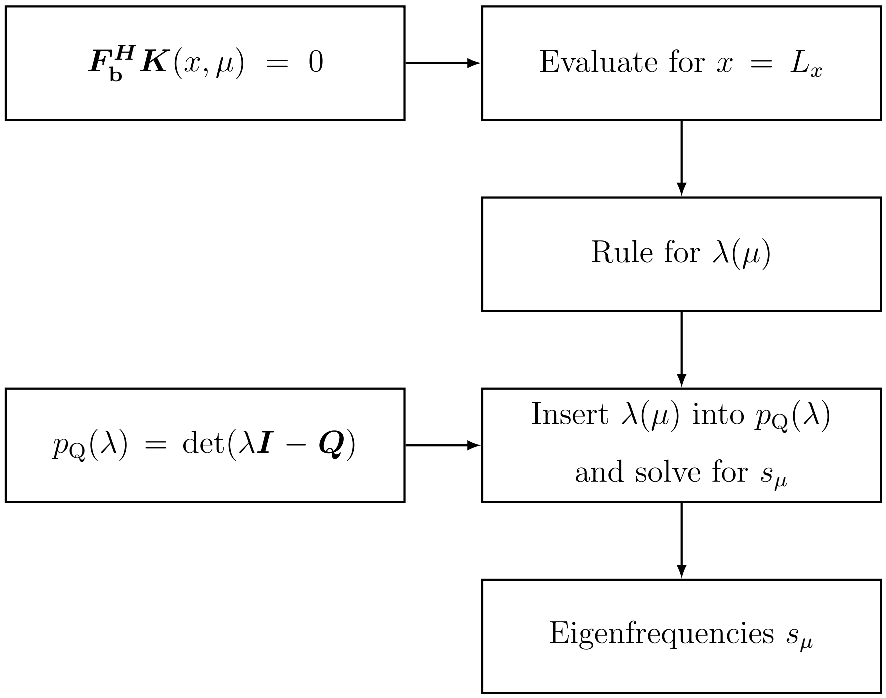


Figure 3.3: Single steps for the calculation of the eigenfrequencies s_μ.

is expanded in the following manner

$$e^{\boldsymbol{Q}\boldsymbol{x}} = \sum_{n=1}^{N} \boldsymbol{F}_n e^{\lambda_n x}. \tag{3.50}$$

The paramter N is defined by the dimension of the matrices in the vector PDE of Eq. (3.1). The coefficient matrix is further divided in the following way

$$\boldsymbol{F}_n = \sum_{n=1}^{N} G_{mn} \boldsymbol{Q}^{m-1}. \tag{3.51}$$

Hereby the matrix $\boldsymbol{G}$ with its single entries is defined as

$$\boldsymbol{G} = \boldsymbol{H}\boldsymbol{V}_0^T \boldsymbol{D'}. \tag{3.52}$$

The matrix $\boldsymbol{H}$ is called the Hankel matrix and depends on the characteristical polynomial $p_Q(\lambda)$. The matrix $\boldsymbol{V}_0^T$ is known as the transposed Vandermonde matrix of the eigenvalues λ. And last, the matrix $\boldsymbol{D'}$ can be developed by the derivatives of the characteristic polynomial

$$\boldsymbol{D'} = \operatorname{diag}\left(\frac{1}{p'_{\mathrm{Q}}(\lambda_1)}, ..., \frac{1}{p'_{\mathrm{Q}}(\lambda_N)}\right). \tag{3.53}$$

This method does not require a calculation of eigenvectors. In general, the method simplifies a complex calculation into manageable substeps. Moreover, Eq. (3.50) shows the impact of the eigenvalues λ on the kernel functions directly.

Chapter 4

Application of the FTM for the PDE of Plates

In the following section the Functional Transformation Method from Sec. 3 is applied to the partial differential equation that was developed in Sec. 2.1. The aim is to obtain the kernel functions, the dispersion relation and the scaling factor for the given PDE.

4.1 Physical Description

For the application of the FTM the subsequent physical representation of plate is chosen. The Eqs. (2.46),(2.27),(2.28) and (2.29) are used and reformulated in this kind of structure

$$\frac{\partial^2 M^{(x)}}{\partial x^2} + 2\frac{\partial^2 M^{(xy)}}{\partial x \partial y} + \frac{\partial^2 M^{(y)}}{\partial y^2} - (\rho h)\frac{\partial^2 w}{\partial t^2} = -p, \tag{4.1}$$

$$-D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) - M^{(x)} = 0, \tag{4.2}$$

$$-D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) - M^{(y)} = 0, \tag{4.3}$$

$$-D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} - M^{(xy)} = 0. \tag{4.4}$$

Note that subscripts of subsequent moments indicate derivations depending on a certain direction. For instance the bending moment $M_x^{(x)}$ means that the specific moment in x-direction is derived with regard to x.

According to Eq. (3.1) the PDE of the plate must be expressed in the following way

$$[\boldsymbol{C}\frac{\partial}{\partial_{tt}} - L]\boldsymbol{y}(\boldsymbol{x},t) = \boldsymbol{f}_e(\boldsymbol{x},t), \qquad \boldsymbol{x} = (x,y), \tag{4.5}$$

whereas the differential operator L is defined as

$$L = \boldsymbol{A} + \nabla\boldsymbol{B}. \tag{4.6}$$

Using the vector $\boldsymbol{y}(\boldsymbol{x},t)$ that is defined as

$$\boldsymbol{y}(\boldsymbol{x},t) = \begin{bmatrix} w \\ M^{(x)} \\ M^{(y)} \\ M^{(xy)} \end{bmatrix}, \tag{4.7}$$

the Eqs. (4.1) - (4.4) can be stated in a new matrix expression

$$\begin{bmatrix} -(\rho h)\,\partial_{tt} & \partial_{xx} & \partial_{yy} & 2\partial_x\partial_y \\ D(\partial_{xx}+\nu\partial_{yy}) & 1 & 0 & 0 \\ D(\partial_{yy}+\nu\partial_{xx}) & 0 & 1 & 0 \\ D(1-\nu)\partial_x\partial_y & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ M^{(x)} \\ M^{(y)} \\ M^{(xy)} \end{bmatrix} = \begin{bmatrix} -p \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.8}$$

The left matrix can be further decomposed into the following parts

$$\partial_{tt}\underbrace{\begin{bmatrix} -\rho h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{C}} - \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{\boldsymbol{A}} - \underbrace{\begin{bmatrix} 0 & -\partial_{xx} & -\partial_{yy} & -2\partial_x\partial_y \\ -D(\partial_{xx}+\nu\partial_{yy}) & 0 & 0 & 0 \\ -D(\partial_{yy}+\nu\partial_{xx}) & 0 & 0 & 0 \\ -D(1-\nu)\partial_x\partial_y & 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{B}\nabla}. \tag{4.9}$$

And again the matrix $\boldsymbol{B}\nabla$ can be rewritten for future calculations as

$$\begin{aligned}\boldsymbol{B}\nabla = \partial_{xx}\begin{bmatrix} 0 & -1 & 0 & 0 \\ -D & 0 & 0 & 0 \\ -D\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \partial_{yy}\begin{bmatrix} 0 & 0 & -1 & 0 \\ -D\nu & 0 & 0 & 0 \\ -D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \partial_{xy}\begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -D(1-\nu) & 0 & 0 & 0 \end{bmatrix} \\ = \partial_{xx}\boldsymbol{B}^{(xx)} + \partial_{yy}\boldsymbol{B}^{(yy)} + \partial_{xy}\boldsymbol{B}^{(xy)}.\end{aligned} \tag{4.10}$$

Based on this equation the differential operator L can be defined as

$$L = \boldsymbol{A} + \partial_{xx}\boldsymbol{B}^{(xx)} + \partial_{yy}\boldsymbol{B}^{(yy)} + \partial_{xy}\boldsymbol{B}^{(xy)}. \tag{4.11}$$

4.2 Initial and Boundary Conditions in the Time Domain

The initial and boundary conditions are defined here according the Eqs. (3.3) and (3.4)

$$\boldsymbol{y}(\boldsymbol{x}, 0) = \boldsymbol{y}_i(\boldsymbol{x}) \qquad x = (x, y), \tag{4.12}$$

$$\boldsymbol{f}_{\mathrm{b}}^{\mathrm{H}}\boldsymbol{y}(\boldsymbol{x}, t) = \boldsymbol{\phi}(\boldsymbol{x}, t) \qquad x = 0, L_x \quad \vee \quad y = 0, L_y. \tag{4.13}$$

The most simple boundary conditions for the plate are the ones for simply supported edges. As in Sec. 2.1.9 already shown several conditions can be established. Fig. 4.1 depicts the relevant boundary conditions for the subsequent calculations. The boundary conditions in Fig. 4.1 are numbered in such a way that later calculations can refer to that definition.

4.3 Laplace Transformation

The first transformation that is applied to the PDE is the Laplace Transformation with respect to Fig. 3.1. Applying the Laplace transform of Eq. (3.5) to the vector PDE of

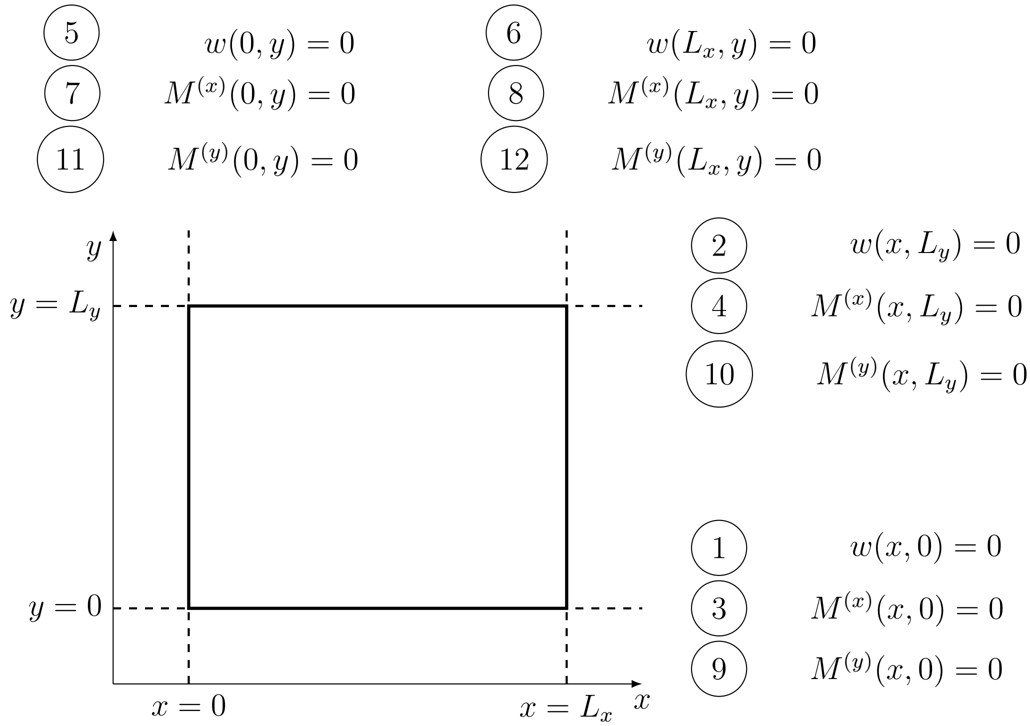


Figure 4.1: Boundary conditions for simply supported edges using moments.

Eq. (4.5) and exploiting the differentiation theorem of Eq. (3.7) leads to

$$\begin{aligned} \left[s^2\boldsymbol{C} - L\right]\boldsymbol{Y}(\boldsymbol{x},s) &= \boldsymbol{F}_{\mathrm{e}}(\boldsymbol{x},s) + s\boldsymbol{y}(\boldsymbol{x},0) + \frac{\partial}{\partial t}\boldsymbol{y}(\boldsymbol{x},0) \\ &= \boldsymbol{F}_{\mathrm{e}}(\boldsymbol{x},s) + s\boldsymbol{y}_{\mathrm{i}}(\boldsymbol{x}) + \frac{\partial}{\partial t}\boldsymbol{y}_{\mathrm{i}}(\boldsymbol{x}). \end{aligned} \tag{4.14}$$

4.4 Boundary Conditions in Temporal Frequency Domain

The boundary conditions are transformed to

$$\boldsymbol{F}_{\mathrm{b}}^{\mathrm{H}}\boldsymbol{Y}(\boldsymbol{x},s) = \boldsymbol{\Phi}(\boldsymbol{x},t), \qquad x = 0, L_x \qquad \vee \qquad y = 0, L_y. \tag{4.15}$$

Knowing those conditions the boundary matrix can be defined along all edges. The underscript bx and by refers to the according edges, where x is set either to 0 or L_x

and analogously for y. Therefore, Eq. (4.15) can be expressed for $x = 0, L_x$ as

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{F}_{\text{bx}}^{\text{H}}} \underbrace{\begin{bmatrix} w(x,y) \\ M^{(x)}(x,y) \\ M^{(y)}(x,y) \\ M^{(xy)}(x,y) \end{bmatrix}}_{\boldsymbol{Y}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\boldsymbol{\Phi}}, \tag{4.16}$$

and for $y = 0, L_y$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{F}_{\text{by}}^{\text{H}}} \underbrace{\begin{bmatrix} w(x,y) \\ M^{(x)}(x,y) \\ M^{(y)}(x,y) \\ M^{(xy)}(x,y) \end{bmatrix}}_{\boldsymbol{Y}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\boldsymbol{\Phi}}. \tag{4.17}$$

From that point of view the vector $\boldsymbol{Y}$ can also be written at the edges as

$$\boldsymbol{Y}(0,y) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ M^{(xy)}(0,y) \end{bmatrix}, \qquad \boldsymbol{Y}(L_x,y) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ M^{(xy)}(L_x,y) \end{bmatrix}, \tag{4.18}$$

$$\boldsymbol{Y}(x,0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ M^{(xy)}(x,0) \end{bmatrix}, \qquad \boldsymbol{Y}(x,L_y) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ M^{(xy)}(x,L_y) \end{bmatrix}. \tag{4.19}$$

4.5 Sturm-Liouville Transformation

As a next step the Sturm-Liouville Transformation is applied to Eq. (4.14). Refering to Eq. (3.11) the following result can be developed

$$s^2\bar{Y}(\mu,s) - \langle L\boldsymbol{Y}(\boldsymbol{x},s), \tilde{\boldsymbol{K}}(\boldsymbol{x},\mu)\rangle = \bar{F}_e(\mu,s) + \bar{y}_i(\mu), \tag{4.20}$$

whereas the scalar product containing the spatial derivatives is still unknown yet. Recalculating the scalar product by inserting L leads to

$$\begin{aligned}&\langle L\boldsymbol{Y}(\boldsymbol{x},s),\tilde{\boldsymbol{K}}(\boldsymbol{x},\mu)\rangle = \langle L\boldsymbol{Y},\tilde{\boldsymbol{K}}\rangle = \\ &\langle \boldsymbol{A}\boldsymbol{Y},\tilde{\boldsymbol{K}}\rangle + \langle \boldsymbol{B}^{(xx)}(\partial_{xx}\boldsymbol{Y}),\tilde{\boldsymbol{K}}\rangle + \langle \boldsymbol{B}^{(yy)}(\partial_{yy}\boldsymbol{Y}),\tilde{\boldsymbol{K}}\rangle + \langle \boldsymbol{B}^{(xy)}(\partial_{xy}\boldsymbol{Y}),\tilde{\boldsymbol{K}}\rangle.\end{aligned} \tag{4.21}$$

From now on this equation is split up and each scalar product is considered individually in order to obtain the boundary terms. The first scalar product including $\boldsymbol{A}$ is known and has not to be rewritten here. Differently, the three right terms including partial derivatives are unknwon. The second term on the right side of Eq. (4.21) is defined as

$$\langle \boldsymbol{B}^{(xx)}(\partial_{xx}\boldsymbol{Y}),\tilde{\boldsymbol{K}}\rangle = \int_V \tilde{\boldsymbol{K}}^{\mathrm{H}}\boldsymbol{B}^{(xx)}(\partial_{xx}\boldsymbol{Y})\mathrm{d}x\mathrm{d}y. \tag{4.22}$$

In order to solve this integral further calculations have to be done. To this end, the rule of integration by parts is applied. This rule is developed by the differentation over a product and can be found in the Appendix C. Applying integration by parts to Eq. (4.22) leads to

$$\begin{aligned}\langle \boldsymbol{B}^{(xx)}\partial_{xx}\boldsymbol{Y},\tilde{\boldsymbol{K}}\rangle = \langle \boldsymbol{B}^{(xx)}\boldsymbol{Y},\partial_{xx}\tilde{\boldsymbol{K}}\rangle &- \int_0^{L_y} \partial_x\tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x,y)\boldsymbol{B}^{(xx)}\boldsymbol{Y}(L_x,y)\mathrm{d}y \\ &+ \int_0^{L_y} \partial_x\tilde{\boldsymbol{K}}^{\mathrm{H}}(0,y)\boldsymbol{B}^{(xx)}\boldsymbol{Y}(0,y)\mathrm{d}y \\ &+ \int_0^{L_y} \tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x,y)\boldsymbol{B}^{(xx)}\partial_x\boldsymbol{Y}(L_x,y)\mathrm{d}y \\ &- \int_0^{L_y} \tilde{\boldsymbol{K}}^{\mathrm{H}}(0,y)\boldsymbol{B}^{(xx)}\partial_x\boldsymbol{Y}(0,y)\mathrm{d}y.\end{aligned} \tag{4.23}$$

Similarly, the term with derivatives in y-direction can be solved

$$\begin{aligned}\langle \boldsymbol{B}^{(yy)}\partial_{yy}\boldsymbol{Y},\tilde{\boldsymbol{K}}\rangle = \langle \boldsymbol{B}^{(yy)}\boldsymbol{Y},\partial_{yy}\tilde{\boldsymbol{K}}\rangle &- \int_0^{L_x} \partial_y\tilde{\boldsymbol{K}}^{\mathrm{H}}(x,L_y)\boldsymbol{B}^{(yy)}\boldsymbol{Y}(x,L_y)\mathrm{d}x \\ &+ \int_0^{L_x} \partial_y\tilde{\boldsymbol{K}}^{\mathrm{H}}(x,0)\boldsymbol{B}^{(yy)}\boldsymbol{Y}(x,0)\mathrm{d}x \\ &+ \int_0^{L_x} \tilde{\boldsymbol{K}}^{\mathrm{H}}(x,L_y)\boldsymbol{B}^{(yy)}\partial_y\boldsymbol{Y}(x,L_y)\mathrm{d}x \\ &- \int_0^{L_x} \tilde{\boldsymbol{K}}^{\mathrm{H}}(x,0)\boldsymbol{B}^{(yy)}\partial_y\boldsymbol{Y}(x,0)\mathrm{d}x.\end{aligned} \tag{4.24}$$

The term including mixed derivatives can also be developed into the following form

$$\begin{aligned}\langle \boldsymbol{B}^{(xy)}\partial_{xy}\boldsymbol{Y},\tilde{\boldsymbol{K}}\rangle = \langle \boldsymbol{B}^{(xy)}\boldsymbol{Y},\partial_{xy}\tilde{\boldsymbol{K}}\rangle &+ \int_0^{L_y} \tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x,y)\boldsymbol{B}^{(xy)}\partial_y\boldsymbol{Y}(L_x,y)\mathrm{d}y \\ &- \int_0^{L_y} \tilde{\boldsymbol{K}}^{\mathrm{H}}(0,y)\boldsymbol{B}^{(xy)}\partial_y\boldsymbol{Y}(0,y)\mathrm{d}y \\ &+ \int_0^{L_x} \tilde{\boldsymbol{K}}^{\mathrm{H}}(x,L_y)\boldsymbol{B}^{(xy)}\partial_x\boldsymbol{Y}(x,L_y)\mathrm{d}x \\ &- \int_0^{L_x} \tilde{\boldsymbol{K}}^{\mathrm{H}}(x,0)\boldsymbol{B}^{(xy)}\partial_x\boldsymbol{Y}(x,0)\mathrm{d}x \\ &- \Big[\tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x,L_y)\boldsymbol{B}^{(xy)}\boldsymbol{Y}(L_x,L_y)\Big] \\ &+ \Big[\tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x,0)\boldsymbol{B}^{(xy)}\boldsymbol{Y}(L_x,0)\Big] \\ &+ \Big[\tilde{\boldsymbol{K}}^{\mathrm{H}}(0,L_y)\boldsymbol{B}^{(xy)}\boldsymbol{Y}(0,L_y)\Big] \\ &- \Big[\tilde{\boldsymbol{K}}^{\mathrm{H}}(0,0)\boldsymbol{B}^{(xy)}\boldsymbol{Y}(0,0)\Big].\end{aligned} \tag{4.25}$$

By means of those calculations Eq. (4.21) can be further expressed as

$$\begin{aligned}\langle L\boldsymbol{Y},\tilde{\boldsymbol{K}}\rangle =& \langle \boldsymbol{A}\boldsymbol{Y},\tilde{\boldsymbol{K}}\rangle + \langle \boldsymbol{B}^{(xx)}\boldsymbol{Y},\partial_{xx}\tilde{\boldsymbol{K}}\rangle + \langle \boldsymbol{B}^{(yy)}\boldsymbol{Y},\partial_{yy}\tilde{\boldsymbol{K}}\rangle + \langle \boldsymbol{B}^{(xy)}\boldsymbol{Y},\partial_{xy}\tilde{\boldsymbol{K}}\rangle \\ &+\Phi_{X_L} + \Phi_{X_0} + \Phi_{Y_L} + \Phi_{Y_0} + \Phi_C,\end{aligned} \tag{4.26}$$

with the boundary terms defined along the edges of the plate

$$\begin{aligned}\Phi_{X_L} = \int_0^{L_y} [&- \partial_x\tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x,y)\boldsymbol{B}^{(xx)}\boldsymbol{Y}(L_x,y) + \tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x,y)\boldsymbol{B}^{(xx)}\partial_x\boldsymbol{Y}(L_x,y) \\ &+ \tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x,y)\boldsymbol{B}^{(xy)}\partial_y\boldsymbol{Y}(L_x,y)]\mathrm{d}y,\end{aligned} \tag{4.27}$$

$$\begin{aligned}\Phi_{X_0} = \int_0^{L_y} [&\partial_x\tilde{\boldsymbol{K}}^{\mathrm{H}}(0,y)\boldsymbol{B}^{(xx)}\boldsymbol{Y}(0,y) - \tilde{\boldsymbol{K}}^{\mathrm{H}}(0,y)\boldsymbol{B}^{(xx)}\partial_x\boldsymbol{Y}(0,y) \\ &- \tilde{\boldsymbol{K}}^{\mathrm{H}}(0,y)\boldsymbol{B}^{(xy)}\partial_y\boldsymbol{Y}(0,y)]\mathrm{d}y,\end{aligned} \tag{4.28}$$

$$\begin{aligned}\Phi_{Y_L} = \int_0^{L_x} [&- \partial_y\tilde{\boldsymbol{K}}^{\mathrm{H}}(x,L_y)\boldsymbol{B}^{(yy)}\boldsymbol{Y}(x,L_y) + \tilde{\boldsymbol{K}}^{\mathrm{H}}(x,L_y)\boldsymbol{B}^{(yy)}\partial_y\boldsymbol{Y}(x,L_y) \\ &+ \tilde{\boldsymbol{K}}^{\mathrm{H}}(x,L_y)\boldsymbol{B}^{(xy)}\partial_x\boldsymbol{Y}(x,L_y)]\mathrm{d}x,\end{aligned} \tag{4.29}$$

$$\begin{aligned}\Phi_{Y_0} = \int_0^{L_x} [&\partial_y\tilde{\boldsymbol{K}}^{\mathrm{H}}(x,0)\boldsymbol{B}^{(yy)}\boldsymbol{Y}(x,0) - \tilde{\boldsymbol{K}}^{\mathrm{H}}(x,0)\boldsymbol{B}^{(yy)}\partial_y\boldsymbol{Y}(x,0) \\ &- \tilde{\boldsymbol{K}}^{\mathrm{H}}(x,0)\boldsymbol{B}^{(xy)}\partial_x\boldsymbol{Y}(x,0)]\mathrm{d}x.\end{aligned} \tag{4.30}$$

According to Eq. (4.26), another term is obtained that provides the following information about the corners of the plate

$$\begin{aligned}\Phi_C = &-\left[\tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x, L_y)\boldsymbol{B}^{(xy)}\boldsymbol{Y}(L_x, L_y)\right] + \left[\tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x, 0)\boldsymbol{B}^{(xy)}\boldsymbol{Y}(L_x, 0)\right] \\ &+ \left[\tilde{\boldsymbol{K}}^{\mathrm{H}}(0, L_y)\boldsymbol{B}^{(xy)}\boldsymbol{Y}(0, L_y)\right] - \left[\tilde{\boldsymbol{K}}^{\mathrm{H}}(0, 0)\boldsymbol{B}^{(xy)}\boldsymbol{Y}(0, 0)\right].\end{aligned} \tag{4.31}$$

4.6 Introduction of a Second Differential Operator

Based on the Eq. (4.26) further calculations lead to

$$\langle L\boldsymbol{Y}, \tilde{\boldsymbol{K}}\rangle = \langle \boldsymbol{A}\boldsymbol{Y}, \tilde{\boldsymbol{K}}\rangle + \langle \boldsymbol{B}^{(xx)}\boldsymbol{Y}, \partial_{xx}\tilde{\boldsymbol{K}}\rangle + \langle \boldsymbol{B}^{(yy)}\boldsymbol{Y}, \partial_{yy}\tilde{\boldsymbol{K}}\rangle + \langle \boldsymbol{B}^{(xy)}\boldsymbol{Y}, \partial_{xy}\tilde{\boldsymbol{K}}\rangle \tag{4.32}$$

$$+\Phi_{X_L} + \Phi_{X_0} + \Phi_{Y_L} + \Phi_{Y_0} + \Phi_C \tag{4.33}$$

$$= \langle \boldsymbol{Y}, \boldsymbol{A}^{\mathrm{H}}\tilde{\boldsymbol{K}}\rangle + \langle \boldsymbol{Y}, \boldsymbol{B}^{(xx)\mathrm{H}}\partial_{xx}\tilde{\boldsymbol{K}}\rangle + \langle \boldsymbol{Y}, \boldsymbol{B}^{(yy)\mathrm{H}}\partial_{yy}\tilde{\boldsymbol{K}}\rangle + \langle \boldsymbol{Y}, \boldsymbol{B}^{(xy)\mathrm{H}}\partial_{xy}\tilde{\boldsymbol{K}}\rangle \tag{4.34}$$

$$+\Phi_{X_L} + \Phi_{X_0} + \Phi_{Y_L} + \Phi_{Y_0} + \Phi_C. \tag{4.35}$$

that can be rewritten in a concise manner

$$\langle L\boldsymbol{Y}, \tilde{\boldsymbol{K}}\rangle = \langle \boldsymbol{Y}, \tilde{L}\tilde{\boldsymbol{K}}\rangle + \Phi_{X_L} + \Phi_{X_0} + \Phi_{Y_L} + \Phi_{Y_0} + \Phi_C, \tag{4.36}$$

whereas $\tilde{L}$ is known as

$$\tilde{L} = \boldsymbol{A}^{\mathrm{H}} + \boldsymbol{B}^{(xx)\mathrm{H}}\partial_{xx} + \boldsymbol{B}^{(yy)\mathrm{H}}\partial_{yy} + \boldsymbol{B}^{(xy)\mathrm{H}}\partial_{xy}. \tag{4.37}$$

4.7 Primal and Adjoint Operator

With regard to the definition of primal and adjoint operators in Sec. 3.5 the following condition has to be fulfilled

$$\langle L\boldsymbol{K}(\boldsymbol{x}, \mu), \tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu)\rangle - \langle \boldsymbol{K}(\boldsymbol{x}, \mu), \tilde{L}\tilde{\boldsymbol{K}}(\boldsymbol{x}, \mu)\rangle = 0 \tag{4.38}$$

and thus it is necessary according to Eq. (4.36) that

$$\Phi_{X_L} + \Phi_{X_0} + \Phi_{Y_L} + \Phi_{Y_0} + \Phi_C = 0. \tag{4.39}$$

In order to fulfill this requirement the following procedure is applied. At first, the scalar product is applied for the vector $\boldsymbol{Y}$ in which the simple boundary conditions are used. Then the resulting terms are set to zero by an appropriate definition for the adjoint kernel. Additionally, Eq. (4.38) is revisited using the kernel $\boldsymbol{K}$ instead of $\boldsymbol{Y}$. From this information knowledge about the kernel can be obtained, so that certain entries of both kernels – the primal and adjoint kernel – can be stated in the end. These boundary terms are considered in the following section in more detail, where it is assumed that each subterm of this superposition equals zero. That assumption is even more strict than necessary, but for the subsequent mathematical calculations more reasonable.

4.8 Boundary Terms

Before investigating the boundary terms in more detail some matrices and definitions of previous sections are recapped. The matrix $\boldsymbol{B}\nabla$ is known from Eq. (4.10)

$$\begin{aligned}\boldsymbol{B}\nabla &= \partial_{xx}\begin{bmatrix} 0 & -1 & 0 & 0 \\ -D & 0 & 0 & 0 \\ -D\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \partial_{yy}\begin{bmatrix} 0 & 0 & -1 & 0 \\ -D\nu & 0 & 0 & 0 \\ -D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \partial_{xy}\begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -D(1-\nu) & 0 & 0 & 0 \end{bmatrix} \\ &= \partial_{xx}\boldsymbol{B}^{(xx)} + \partial_{yy}\boldsymbol{B}^{(yy)} + \partial_{xy}\boldsymbol{B}^{(xy)}.\end{aligned} \tag{4.40}$$

The vectors $\boldsymbol{Y}$ and $\boldsymbol{K}$ are defined as

$$\boldsymbol{Y}(x,y) = \begin{bmatrix} w(x,y) \\ M^{(x)}(x,y) \\ M^{(y)}(x,y) \\ M^{(xy)}(x,y) \end{bmatrix}, \qquad \boldsymbol{K}(x,y) = \begin{bmatrix} K_1(x,y) \\ K_2(x,y) \\ K_3(x,y) \\ K_4(x,y) \end{bmatrix}. \tag{4.41}$$

Now the following matrix combinations are calculated

$$\boldsymbol{B}^{(xx)}\boldsymbol{Y}(x,y) = \begin{bmatrix} -M^{(x)}(x,y) \\ -Dw(x,y) \\ -D\nu w(x,y) \\ 0 \end{bmatrix}, \qquad \boldsymbol{B}^{(xx)}\boldsymbol{K}(x,y) = \begin{bmatrix} -K_2(x,y) \\ -DK_1(x,y) \\ -D\nu K_1(x,y) \\ 0 \end{bmatrix}, \tag{4.42}$$

$$\boldsymbol{B}^{(yy)}\boldsymbol{Y}(x,y) = \begin{bmatrix} -M^{(y)}(x,y) \\ -D\nu w(x,y) \\ -Dw(x,y) \\ 0 \end{bmatrix}, \qquad \boldsymbol{B}^{(yy)}\boldsymbol{K}(x,y) = \begin{bmatrix} -K_3(x,y) \\ -D\nu K_1(x,y) \\ -DK_1(x,y) \\ 0 \end{bmatrix}, \tag{4.43}$$

$$\boldsymbol{B}^{(xy)}\boldsymbol{Y}(x,y) = \begin{bmatrix} -2M^{(xy)}(x,y) \\ 0 \\ 0 \\ -D(1-\nu)w(x,y) \end{bmatrix}, \qquad \boldsymbol{B}^{(xy)}\boldsymbol{K}(x,y) = \begin{bmatrix} -2K_4(x,y) \\ 0 \\ 0 \\ -D(1-\nu)K_1(x,y) \end{bmatrix}. \tag{4.44}$$

The boundary terms are expressed according to Eq. (4.27). At first the term $\Phi^{\mathrm{Y}}_{X_L}$ from Eq. (4.39) is considered, where the subscript X_L describes the edge of the plate for $x = L_x$. The superscript Y denotes that the scalar product is applied to the vector of $\boldsymbol{Y}(x,y)$

$$\begin{aligned} \Phi^{\mathrm{Y}}_{X_L} = \int_0^{L_y} [&\underbrace{-\partial_x \tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x,y)\boldsymbol{B}^{(xx)}\boldsymbol{Y}(L_x,y)}_{(a)} \underbrace{+\tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x,y)\boldsymbol{B}^{(xx)}\partial_x\boldsymbol{Y}(L_x,y)}_{(b)} \\ &\underbrace{+\tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x,y)\boldsymbol{B}^{(xy)}\partial_y\boldsymbol{Y}(L_x,y)}_{(c)}]\mathrm{d}y = 0. \end{aligned} \tag{4.45}$$

The mentioned term $\Phi^{\mathrm{Y}}_{X_L}$ must result in zero. For reasons of simplified calculations an even stricter requirement is applied here by setting the subterms $(a), (b), (c)$ to zero. This assumption is valid due to the fact that an integral over zero is zero. Using the information of boundary conditions of Eq. (4.19) and Eq. (4.42) it can be followed that

$$\boldsymbol{B}^{(xx)}\boldsymbol{Y}(L_x,y) = 0, \tag{4.46}$$

and thus the part (a) vanishes. Contrarily to this, the part (b) contains derivatives applied on $\boldsymbol{Y}$, which leads to the product

$$\boldsymbol{B}^{(xx)}\boldsymbol{Y}_x(L_x, y) = \begin{bmatrix} -M_x^{(x)}(L_x, y) \\ -Dw_x(L_x, y) \\ -D\nu w_x(L_x, y) \\ 0 \end{bmatrix}. \tag{4.47}$$

This term disappears if the adjoint kernel is structured like

$$\tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x, y) = \begin{bmatrix} \tilde{K}_1^*(L_x, y) \\ \tilde{K}_2^*(L_x, y) \\ \tilde{K}_3^*(L_x, y) \\ \tilde{K}_4^*(L_x, y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \tilde{K}_4^*(L_x, y) \end{bmatrix}. \tag{4.48}$$

From the term (c) follows by the derivative in y-direction

$$\boldsymbol{B}^{(xy)}\boldsymbol{Y}_y(L_x, y) = \begin{bmatrix} -2M_y^{(xy)}(L_x, y) \\ 0 \\ 0 \\ -D(1-\nu)\underbrace{w_y(L_x, y)}_{=0} \end{bmatrix} = \begin{bmatrix} -2M_y^{(xy)}(L_x, y) \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.49}$$

Therefore, it is necessary to demand $\tilde{K}_1^*(L_x, y) = 0$ in order to cancel the term $\boldsymbol{B}^{(xy)}\boldsymbol{Y}_y(L_x, y)$. However, this is already be fulfilled by the definition in Eq. (4.48). So it can be summed up that the first boundary term $\Phi_{X_L}^{\mathrm{Y}}$ vanishes if the adjoint kernel at $x = L_x$ is defined as

$$\tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x, y) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \tilde{K}_4^*(L_x, y) \end{bmatrix}. \tag{4.50}$$

Next the boundary term of the kernel function are considered. Analogously to Eq. (4.45), the following equation can be established

$$\Phi^{\mathrm{K}}_{X_L} = \int_0^{L_y} [\underbrace{-\tilde{\boldsymbol{K}}_x^{\mathrm{H}}(L_x,y)\boldsymbol{B}^{(xx)}\boldsymbol{K}(L_x,y)}_{(a)} \underbrace{+\tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x,y)\boldsymbol{B}^{(xx)}\boldsymbol{K}_x(L_x,y)}_{(b)} \tag{4.51}$$

$$\underbrace{+\tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x,y)\boldsymbol{B}^{(xy)}\boldsymbol{K}_y(L_x,y)}_{(c)}]\mathrm{d}y. \tag{4.52}$$

Again here the named terms of the superposititon are considererd individually. Term (a) leads to 0 due to the following matrix multiplication

$$\tilde{\boldsymbol{K}}_x^{\mathrm{H}}(L_x,y)\boldsymbol{B}^{(xx)} = \begin{bmatrix}0 & 0 & 0 & \partial_x\tilde{K}_4^*(L_x,y)\end{bmatrix}\begin{bmatrix}0 & 0 & -1 & 0\\ -D\nu & 0 & 0 & 0\\ -D & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} = 0. \tag{4.53}$$

Similarly to term (a), also term (b) vanishes due to

$$\tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x,y)\boldsymbol{B}^{(xx)} = 0. \tag{4.54}$$

The term (c) leads to the following multiplication

$$\tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x,y)\boldsymbol{B}^{(xy)} = \begin{bmatrix}0 & 0 & 0 & \tilde{K}_4^*(L_x,y)\end{bmatrix}\begin{bmatrix}0 & 0 & 0 & -2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -D(1-\nu) & 0 & 0 & 0\end{bmatrix} \tag{4.55}$$

$$= \begin{bmatrix}-\tilde{K}_4^*(L_x,y)D(1-\nu) & 0 & 0 & 0\end{bmatrix}. \tag{4.56}$$

Therefore, the complete term (c) results in

$$\tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x,y)\boldsymbol{B}^{(xy)}\boldsymbol{K}_y(L_x,y) = -\tilde{K}_4^*(L_x,y)D(1-\nu)\partial_y K_1(L_x,y). \tag{4.57}$$

Eq. (4.57) is zero if the first entry of the primal kernel function is zero at the boundary $x = L_x$

$$K_1(L_x,y) = 0. \tag{4.58}$$

That information allows now to define a boundary matrix $\boldsymbol{F}^{\mathrm{H}}_{bX_L}$ as

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{F}^{\mathrm{H}}_{bX_L}} \begin{bmatrix} K_1(L_x, y) \\ K_2(L_x, y) \\ K_3(L_x, y) \\ K_4(L_x, y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.59}$$

Referring to the structure of $\tilde{\boldsymbol{K}}$ at $x = L_x$ (see Eq (4.50)), the boundary matrix $\tilde{\boldsymbol{F}}^{\mathrm{H}}_{bX_L}$ is represented as

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\tilde{\boldsymbol{F}}^{\mathrm{H}}_{bX_L}} \begin{bmatrix} \tilde{K}^*_1(L_x, y) \\ \tilde{K}^*_2(L_x, y) \\ \tilde{K}^*_3(L_x, y) \\ \tilde{K}^*_4(L_x, y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.60}$$

The calculation was shown here for the boundary term Φ_{X_L}, it can be similiarly done for the other boundary terms: Φ_{Y_L}, Φ_{X_0} and Φ_{Y_0}. For the boundary term Φ_{Y_L} the following boundary matrices can be derived

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{F}^{\mathrm{H}}_{bX_0}} \begin{bmatrix} K_1(0, y) \\ K_2(0, y) \\ K_3(0, y) \\ K_4(0, y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\tilde{\boldsymbol{F}}^{\mathrm{H}}_{bX_0}} \begin{bmatrix} \tilde{K}^*_1(0, y) \\ \tilde{K}^*_2(0, y) \\ \tilde{K}^*_3(0, y) \\ \tilde{K}^*_4(0, y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.61}$$

According to the boundary term of Φ_{X_0} the following matrices can be obtained

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{F}^{\mathrm{H}}_{bY_L}} \begin{bmatrix} K_1(x, L_y) \\ K_2(x, L_y) \\ K_3(x, L_y) \\ K_4(x, L_y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\tilde{\boldsymbol{F}}^{\mathrm{H}}_{bY_L}} \begin{bmatrix} \tilde{K}^*_1(x, L_y) \\ \tilde{K}^*_2(x, L_y) \\ \tilde{K}^*_3(x, L_y) \\ \tilde{K}^*_4(x, L_y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.62}$$

The last boundary term Φ_{Y_0} leads to boundary matrices of the form

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{F}^{\mathrm{H}}_{bY_0}} \begin{bmatrix} K_1(x, 0) \\ K_2(x, 0) \\ K_3(x, 0) \\ K_4(x, 0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\tilde{\boldsymbol{F}}^{\mathrm{H}}_{bY_0}} \begin{bmatrix} \tilde{K}^*_1(x, 0) \\ \tilde{K}^*_2(x, 0) \\ \tilde{K}^*_3(x, 0) \\ \tilde{K}^*_4(x, 0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.63}$$

Now all boundary terms along the edges of the plate are solved and the boundary matrices are developed. The term Φ_C defining the corners of the plate is still unprocessed. It is known from the Eq. (4.31) that Φ_C is

$$\begin{aligned} \Phi_C = & \underbrace{-\left[\tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x, L_y)\boldsymbol{B}^{(xy)}\boldsymbol{Y}(L_x, L_y)\right]}_{\Phi_{C_1}} \underbrace{+\left[\tilde{\boldsymbol{K}}^{\mathrm{H}}(L_x, 0)\boldsymbol{B}^{(xy)}\boldsymbol{Y}(L_x, 0)\right]}_{\Phi_{C_2}} \\ & \underbrace{+\left[\tilde{\boldsymbol{K}}^{\mathrm{H}}(0, L_y)\boldsymbol{B}^{(xy)}\boldsymbol{Y}(0, L_y)\right]}_{\Phi_{C_3}} \underbrace{-\left[\tilde{\boldsymbol{K}}^{\mathrm{H}}(0, 0)\boldsymbol{B}^{(xy)}\boldsymbol{Y}(0, 0)\right]}_{\Phi_{C_4}}. \end{aligned} \tag{4.64}$$

The subterms of Φ_C result in

$$\begin{aligned} \Phi_{C_1} &= \tilde{K}^*_1(L_x, L_y)2M^{(xy)}(L_x, L_y) + \tilde{K}^*_4(L_x, L_y)D(1-\nu)w(L_x, L_y), \\ \Phi_{C_2} &= -\tilde{K}^*_1(L_x, 0)2M^{(xy)}(L_x, 0) - \tilde{K}^*_4(L_x, 0)D(1-\nu)w(L_x, 0), \\ \Phi_{C_3} &= -\tilde{K}^*_1(0, L_y)2M^{(xy)}(0, L_y) - \tilde{K}^*_4(0, L_y)D(1-\nu)w(0, L_y), \\ \Phi_{C_4} &= \tilde{K}^*_1(0, 0)2M^{(xy)}(0, 0) + \tilde{K}^*_4(0, 0)D(1-\nu)w(0, 0). \end{aligned} \tag{4.65}$$

Referring to the boundary conditions of Eq. (2.57) it is known that $w(x, y) = 0$ for $x = 0, L_x$ and $y = 0, L_y$. Using this and the Eqs. (4.60) - (4.63) it follows

$$\Phi_{C_1} = \Phi_{C_2} = \Phi_{C_3} = \Phi_{C_4} = 0. \tag{4.66}$$

Regarding to those equations the boundary terms for the corners are cancelled

$$\Rightarrow \qquad \Phi_C = 0. \tag{4.67}$$

As a summary the subsequent Table 4.1 offers an overview for the boundary terms and its kernel conditions for simply supported edges.

Table 4.1: Results and Conditions for the Boundary Terms.

$\boldsymbol{\Phi^{\mathrm{K}}_{X_L} = 0}$	$\boldsymbol{\Phi^{\mathrm{K}}_{x_0} = 0}$	$\boldsymbol{\Phi^{\mathrm{K}}_{Y_L} = 0}$	$\boldsymbol{\Phi^{\mathrm{K}}_{Y_0} = 0}$	$\boldsymbol{\Phi_C = 0}$
$K_1(L_x, y) = 0$	$K_1(0, y) = 0$	$K_1(x, L_y) = 0$	$K_1(x, 0) = 0$	$\tilde{K}_1^*(L_x, L_y) = 0$
$\tilde{K}_1^*(L_x, y) = 0$	$\tilde{K}_1^*(0, y) = 0$	$\tilde{K}_1^*(x, L_y) = 0$	$\tilde{K}_1^*(x, 0) = 0$	$\tilde{K}_1^*(L_x, 0) = 0$
$\tilde{K}_2^*(L_x, y) = 0$	$\tilde{K}_2^*(0, y) = 0$	$\tilde{K}_2^*(x, L_y) = 0$	$\tilde{K}_2^*(x, 0) = 0$	$\tilde{K}_1^*(0, L_y) = 0$
$\tilde{K}_3^*(L_x, y) = 0$	$\tilde{K}_3^*(0, y) = 0$	$\tilde{K}_3^*(x, L_y) = 0$	$\tilde{K}_3^*(x, 0) = 0$	$\tilde{K}_1^*(0, 0) = 0$

4.9 Eigenvalue Problem

This section deals with the eigenvalue problem that can be defined for the primal and adjoint kernel.

4.9.1 Primal Kernel

The eigenvalue problem for the primal kernel is defined by Eq. (3.27). For the application to the plate equation, the spatial frequencies s_μ are chosen to be quadratic in the eigenvalue problem, because of the second order derivatives in the differential operator L

$$L\boldsymbol{K}(\boldsymbol{x}, \mu) = s_\mu^2 \boldsymbol{C}\boldsymbol{K}(\boldsymbol{x}, \mu), \qquad \boldsymbol{x} \in V. \tag{4.68}$$

With the differential operator L from Eq. (4.11) it follows for the eigenvalue problem

$$\left(\boldsymbol{A} + \partial_{xx}\boldsymbol{B}^{(xx)} + \partial_{yy}\boldsymbol{B}^{(yy)} + \partial_{xy}\boldsymbol{B}^{(xy)}\right)\boldsymbol{K}(\boldsymbol{x},\mu) = s_\mu^2\boldsymbol{C}\boldsymbol{K}(\boldsymbol{x},\mu), \tag{4.69}$$

$$\left(\partial_{xx}\boldsymbol{B}^{(xx)} + \partial_{yy}\boldsymbol{B}^{(yy)} + \partial_{xy}\boldsymbol{B}^{(xy)}\right)\boldsymbol{K}(\boldsymbol{x},\mu) = \left(s_\mu^2\boldsymbol{C} - \boldsymbol{A}\right)\boldsymbol{K}(\boldsymbol{x},\mu). \tag{4.70}$$

Exploiting the matrix structures from Eqs. (4.9), (4.10) the following set of equations is developed

$$-\partial_{xx}K_2(\boldsymbol{x},\mu) - \partial_{yy}K_3(\boldsymbol{x},\mu) - 2\partial_{xy}K_4(\boldsymbol{x},\mu) = -s_\mu^2\rho h K_1(\boldsymbol{x},\mu), \tag{4.71}$$

$$K_2(\boldsymbol{x},\mu) = -D\left[\partial_{xx} + \nu\partial_{yy}\right]K_1(\boldsymbol{x},\mu), \tag{4.72}$$

$$K_3(\boldsymbol{x},\mu) = -D\left[\nu\partial_{xx} + \partial_{yy}\right]K_1(\boldsymbol{x},\mu), \tag{4.73}$$

$$K_4(\boldsymbol{x},\mu) = -D(1-\nu)\partial_{xy}K_1(\boldsymbol{x},\mu). \tag{4.74}$$

Inserting Eqs. (4.72) - (4.74) into (4.71) leads to

$$\begin{aligned}&\partial_{xx}D\left[\partial_{xx} + \nu\partial_{yy}\right]K_1(\boldsymbol{x},\mu) + \partial_{yy}D\left[\nu\partial_{xx} + \partial_{yy}\right]K_1(\boldsymbol{x},\mu) + 2\partial_{xy}D(1-\nu)\partial_{xy}K_1(\boldsymbol{x},\mu)\\ &= -s_\mu^2\rho h K_1(\boldsymbol{x},\mu),\end{aligned} \tag{4.75}$$

which can be simplified to

$$s_\mu^2 K_1(\boldsymbol{x},\mu) = \frac{-D}{\rho h}\left[\partial_{xxxx} + 2\partial_{xxyy} + \partial_{yyyy}\right]K_1(\boldsymbol{x},\mu) \tag{4.76}$$

$$= \frac{-D}{\rho h}\left[\partial_{xx} + \partial_{yy}\right]^2 K_1(\boldsymbol{x},\mu). \tag{4.77}$$

4.9.2 Adjoint Kernel

Analogously, the eigenvalue problem can be formulated for the adjoint kernel (see Eq. (3.26))

$$\tilde{L}\tilde{\boldsymbol{K}}(\boldsymbol{x},\mu) = s_\mu^{*2}\boldsymbol{C}^{\mathrm{H}}\tilde{\boldsymbol{K}}(\boldsymbol{x},\mu), \qquad \boldsymbol{x} \in V, \tag{4.78}$$

whereas $\tilde{L}$ is defined as

$$\tilde{L} = \boldsymbol{A}^{\mathrm{H}} + \boldsymbol{B}^{(xx)\mathrm{H}}\partial_{xx} + \boldsymbol{B}^{(yy)\mathrm{H}}\partial_{yy} + \boldsymbol{B}^{(xy)\mathrm{H}}\partial_{xy}. \tag{4.79}$$

Based on that adjoint operator the following equation must be solved

$$\left(\partial_{xx}\boldsymbol{B}^{(xx)\mathrm{H}} + \partial_{yy}\boldsymbol{B}^{(yy)\mathrm{H}} + \partial_{xy}\boldsymbol{B}^{(xy)\mathrm{H}}\right)\tilde{\boldsymbol{K}}(\boldsymbol{x},\mu) = \left(s_\mu^{*2}\boldsymbol{C}^{\mathrm{H}} - \boldsymbol{A}^{\mathrm{H}}\right)\tilde{\boldsymbol{K}}(\boldsymbol{x},\mu). \quad (4.80)$$

The following hermetic matrices can be calculated with regard to the Eqs. (4.9) and (4.10)

$$\partial_{xx}\boldsymbol{B}^{(xx)\mathrm{H}} + \partial_{yy}\boldsymbol{B}^{(yy)\mathrm{H}} + \partial_{xy}\boldsymbol{B}^{(xy)\mathrm{H}} = \begin{bmatrix} 0 & -D(\partial_{xx}+\nu\partial_{yy}) & -D(\partial_{yy}+\nu\partial_{xx}) & -D(1-\nu)\partial_x\partial_y \\ -\partial_{xx} & 0 & 0 & 0 \\ -\partial_{yy} & 0 & 0 & 0 \\ -2\partial_x\partial_y & 0 & 0 & 0 \end{bmatrix}, \quad (4.81)$$

$$s_\mu^{*2}\boldsymbol{C}^{\mathrm{H}} - \boldsymbol{A}^{\mathrm{H}} = \begin{bmatrix} -s_\mu^{*2}\rho h & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (4.82)$$

Now the following equations can be developed

$$-s_\mu^{*2}\rho h\tilde{K}_1(\boldsymbol{x},\mu) = -D\left[\partial_{xx}+\nu\partial_{yy}\right]\tilde{K}_2(\boldsymbol{x},\mu) - D\left[\nu\partial_{xx}+\partial_{yy}\right]\tilde{K}_3 - D(1-\nu)\partial_{xy}K_4(\boldsymbol{x},\mu), \quad (4.83)$$

$$\tilde{K}_2(\boldsymbol{x},\mu) = -\partial_{xx}\tilde{K}_1(\boldsymbol{x},\mu), \quad (4.84)$$

$$\tilde{K}_3(\boldsymbol{x},\mu) = -\partial_{yy}\tilde{K}_1(\boldsymbol{x},\mu), \quad (4.85)$$

$$\tilde{K}_4(\boldsymbol{x},\mu) = -2\partial_{xy}\tilde{K}_1(\boldsymbol{x},\mu). \quad (4.86)$$

Substituting the kernels of Eqs. (4.84) - (4.86) in Eq. (4.83) leads to

$$D\left[\partial_{xx}+\nu\partial_{yy}\right]\partial_{xx}\tilde{K}_1(\boldsymbol{x},\mu) + D\left[\nu\partial_{xx}+\partial_{yy}\right]\partial_{yy}\tilde{K}_1(\boldsymbol{x},\mu) + D(1-\nu)\partial_{xy}2\partial_{xy}\tilde{K}_1(\boldsymbol{x},\mu) = -s_\mu^{*2}\rho h\tilde{K}_1(\boldsymbol{x},\mu), \quad (4.87)$$

which can be simplified to

$$s_\mu^{*2}\tilde{K}_1(\boldsymbol{x},\mu) = \frac{-D}{\rho h}\left[\partial_{xxxx} + 2\partial_{xxyy} + \partial_{yyyy}\right]\tilde{K}_1(\boldsymbol{x},\mu) \tag{4.88}$$

$$= \frac{-D}{\rho h}\left[\partial_{xx} + \partial_{yy}\right]^2\tilde{K}_1(\boldsymbol{x},\mu). \tag{4.89}$$

4.10 Kernel Functions

The kernel functions are developed for the primal and adjoint kernel. For the development of kernel functions it is necessary to assume a specific solution for $K_1(\boldsymbol{x},\mu)$, since the system of equations in Sec. 4.9 can not be further solved.

4.10.1 Primal Kernel

With regard to the boundary conditions of Fig. 4.1 and the definition of the kernel at the boundaries of Eqs. (4.59) - (4.63) the subsequent solution is assumed

$$K_1(\boldsymbol{x},\mu) = \sin(\lambda_x x)\sin(\lambda_y y) = \sin\left(\kappa_x\frac{\pi}{L_x}x\right)\sin\left(\kappa_y\frac{\pi}{L_y}y\right), \tag{4.90}$$

with the wave numbers κ_x in x-direction and κ_y in y-direction. Its derivatives can be calculated as

$$\frac{\partial^2}{\partial x^2}K_1(\boldsymbol{x},\mu) = -\lambda_x^2 K_1(\boldsymbol{x},\mu), \tag{4.91}$$

$$\frac{\partial^2}{\partial y^2}K_1(\boldsymbol{x},\mu) = -\lambda_y^2 K_1(\boldsymbol{x},\mu). \tag{4.92}$$

It can be seen that the Eqs. (4.90) - (4.92) equals zero for $x = 0, L_x$ and $y = 0, L_y$. Therefore, the boundary conditions of Fig. 4.1 are fulfilled under the assumption of Eq. (4.90). The defintion of a combination of two sine waves is thus valid. The solution of Eq. (4.76) requires further derivatives

$$\frac{\partial^4}{\partial x^4}K_1(\boldsymbol{x},\mu) = \lambda_x^4 K_1(\boldsymbol{x},\mu), \qquad \frac{\partial^4}{\partial y^4}K_1(\boldsymbol{x},\mu) = \lambda_x^4 K_1(\boldsymbol{x},\mu), \tag{4.93}$$

$$\frac{\partial^2}{\partial xy} K_1(\boldsymbol{x}, \mu) = \lambda_x \lambda_y \cos(\lambda_x x) \cos(\lambda_y y), \tag{4.94}$$

$$\Rightarrow \qquad \frac{\partial^4}{\partial x^2 y^2} K_1(\boldsymbol{x}, \mu) = \lambda_x^2 \lambda_y^2 K_1(\boldsymbol{x}, \mu). \tag{4.95}$$

The primal kernel matrix can now be calculated with regard to the Eqs. (4.72),(4.73) and (4.74)

$$\boldsymbol{K}(\boldsymbol{x}, \mu) \begin{bmatrix} \sin(\lambda_x x) \sin(\lambda_y y) \\ D\left[\lambda_x^2 + \kappa \lambda_y^2\right] \sin(\lambda_x x) \sin(\lambda_y y) \\ D\left[\lambda_y^2 + \kappa \lambda_x^2\right] \sin(\lambda_x x) \sin(\lambda_y y) \\ -D\lambda_x \lambda_y (1-\kappa) \cos(\lambda_x x) \cos(\lambda_y y) \end{bmatrix}. \tag{4.96}$$

4.10.2 Adjoint Kernel

In order to calculate the adjoint kernel the following assumption is set analogously to the primal kernel

$$\tilde{K}_1(\boldsymbol{x}, \mu) = \sin(\lambda_x x) \sin(\lambda_y y) = \sin\left(\kappa_x \frac{\pi}{L_x} x\right) \sin\left(\kappa_y \frac{\pi}{L_y} y\right), \tag{4.97}$$

with the wave numbers κ_x in x-direction and κ_y in y-direction. Reusing the Eqs. (4.84) - (4.86) the following results for the adjoint kernel can developed

$$\begin{aligned} \tilde{K}_2(\boldsymbol{x}, \mu) &= -\partial_{xx} \tilde{K}_1(\boldsymbol{x}, \mu) \\ &= \lambda_x^2 \left[\sin(\lambda_x x) \sin(\lambda_y y)\right], \end{aligned} \tag{4.98}$$

$$\begin{aligned} \tilde{K}_3(\boldsymbol{x}, \mu) &= -\partial_{yy} \tilde{K}_1(\boldsymbol{x}, \mu) \\ &= \lambda_y^2 \left[\sin(\lambda_x x) \sin(\lambda_y y)\right], \end{aligned} \tag{4.99}$$

$$\begin{aligned} \tilde{K}_4(\boldsymbol{x}, \mu) &= -2\partial_{xy} \tilde{K}_1(\boldsymbol{x}, \mu) \\ &= -2\lambda_x \lambda_y \left[\cos(\lambda_x x) \cos(\lambda_y y)\right]. \end{aligned} \tag{4.100}$$

Therefore, the adjoint kernel matrix can be evaluated as

$$\tilde{\boldsymbol{K}}(\boldsymbol{x},\mu) = \begin{bmatrix} \sin(\lambda_x x)\sin(\lambda_y y) \\ \lambda_x^2\left[\sin(\lambda_x x)\sin(\lambda_y y)\right] \\ \lambda_y^2\left[\sin(\lambda_x x)\sin(\lambda_y y)\right] \\ -2\lambda_x\lambda_y\left[\cos(\lambda_x x)\cos(\lambda_y y)\right] \end{bmatrix}. \tag{4.101}$$

4.11 Dispersion Relation

Based on Eqs. (4.93) - (4.95) the Eq. (4.76) can be reformulated as

$$s_\mu^2 = \frac{-D}{\rho h}\left[\lambda_x^4 + 2\lambda_x^2\lambda_y^2 + \lambda_y^4\right] \tag{4.102}$$

$$= \frac{-D}{\rho h}\left[\lambda_x^2 + \lambda_y^2\right]^2. \tag{4.103}$$

This equation leads to the "Dispersion Relation" and shows the connection between the temporal and spatial frequencies

$$\Rightarrow s_\mu = \pm j\sqrt{\frac{D}{\rho h}}\left[\lambda_x^2 + \lambda_y^2\right]. \tag{4.104}$$

Furthermore, the dispersion relation can be developed by means of the adjoint eigenvalue problem. Similarly, based on the Eqs. (4.98) - (4.100) the Eq. (4.88) can be rewritten as

$$s_\mu^{*2} = \frac{-D}{\rho h}\left[\lambda_x^2 + \lambda_y^2\right]^2, \tag{4.105}$$

$$\Rightarrow s_\mu^* = \pm j\sqrt{\frac{D}{\rho h}}\left[\lambda_x^2 + \lambda_y^2\right]. \tag{4.106}$$

The dimension of the eigenvalue s_μ is shown in the Appendix D.1.

4.12 Scaling Factor

According to Eq. (3.39) the scaling factor can be calculated by the scalar product

$$\langle \boldsymbol{C}\boldsymbol{K}_\mu, \tilde{\boldsymbol{K}}_\kappa\rangle = \int_V \tilde{\boldsymbol{K}}^{\mathrm{H}}\boldsymbol{C}\boldsymbol{K}(\boldsymbol{x},s)\mathrm{d}x\mathrm{d}y. \tag{4.107}$$

The biorthogonality is hereby defined together with the matrix $\boldsymbol{C}$. As a first step the inner matrix multiplication is investigated in more detail

$$\tilde{\boldsymbol{K}}^{\mathrm{H}}\boldsymbol{C}\boldsymbol{K}(\boldsymbol{x},s). \tag{4.108}$$

Using the definitions of the primal and adjoint kernel from Eqs. (4.96),(4.101) and the matrix $\boldsymbol{C}$ of Eq. (4.9) it follows for the product

$$\begin{aligned}
\tilde{\boldsymbol{K}}^{\mathrm{H}}\boldsymbol{C}\boldsymbol{K}(\boldsymbol{x},s) &= \begin{bmatrix}-\rho h\sin(\lambda_x x)\sin(\lambda_y y) & 0 & 0 & 0\end{bmatrix}\boldsymbol{K}(\boldsymbol{x},s)\\
&= -\rho h\sin(\lambda_x x)\sin(\lambda_y y)\sin(\lambda_x x)\sin(\lambda_y y)\\
&= -\rho h\sin^2(\lambda_x x)\sin^2(\lambda_y y)\\
&= -\rho h\sin^2(\kappa_x\frac{\pi}{L_x}x)\sin^2(\kappa_y\frac{\pi}{L_y}y).
\end{aligned} \tag{4.109}$$

Consequently, the integral in Eq. (4.107) results in

$$\int_0^{L_y}\int_0^{L_x} -\rho h\sin^2(\kappa_x\frac{\pi}{L_x}x)\sin^2(\kappa_y\frac{\pi}{L_y}y)\mathrm{d}x\mathrm{d}y. \tag{4.110}$$

The application of two integrals can be done separately. The integral over x follows as

$$\int_0^{L_x} -\rho h\sin^2(\kappa_x\frac{\pi}{L_x}x)\sin^2(\kappa_y\frac{\pi}{L_y}y)\mathrm{d}x \tag{4.111}$$

$$= \left[-\rho h\left[\frac{x}{2}-\frac{L_x}{4\kappa_x\pi}\sin\left(2\kappa_x\frac{\pi}{L_x}x\right)\right]\sin^2(\kappa_y\frac{\pi}{L_y}y)\right]_0^{L_x} \tag{4.112}$$

$$= -\rho h\frac{L_x}{2}\sin^2(\kappa_y\frac{\pi}{L_y}), \tag{4.113}$$

by using the following rule for the integral $\int_0^{L_x} sin^2(ax)\mathrm{d}x = \left[\frac{x}{2}-\frac{1}{4a}\sin(2ax)\right]_0^{L_x}$. Applying that integration rule over y leads to

$$\int_0^{L_y} -\rho h\frac{L_x}{2}\sin^2(\kappa_y\frac{\pi}{L_y})\mathrm{d}y = \left[-\rho h\frac{L_x}{2}\left[\frac{y}{2}-\frac{L_y}{4\kappa_y\pi}\sin\left(2\kappa_y\frac{\pi}{L_y}y\right)\right]\right]_0^{L_y} \tag{4.114}$$

$$= -\rho h\frac{L_x}{2}\frac{L_y}{2}. \tag{4.115}$$

Thus the scaling factor can be calculated according to Eq. (3.39) as

$$N_\mu = -\frac{\rho h}{4}L_xL_y. \tag{4.116}$$

Chapter 5

Comparison to the Navier's Solution

In the following chapter the developed results are compared to the Navier's solution for simply supported vibrating plates (see Sec. 2.2.3). Due to the fact that the Navier's solution considers a static, not time-dependent behaviour the developed solution of the FTM had to be reformulated for the static case. The following calculation shows a possible approach of a comparison.
To this end, Eq. (3.41) is recapped

$$\boldsymbol{Y}(\boldsymbol{x}, s) = \sum_{\mu} \frac{1}{N_\mu} \bar{Y}(\mu, s) \boldsymbol{K}(\boldsymbol{x}, \mu), \tag{5.1}$$

whereas $\bar{Y}(\mu, s)$ is known as

$$\bar{Y}(\mu, s) = \frac{1}{s^2 - s_\mu^2} \left[\bar{F}_\mathrm{e}(\mu, s) + \bar{y}_i(\mu) - \phi_{x_L} - \phi_{x_0} - \phi_{y_L} - \phi_{y_0} - \phi_c \right] \tag{5.2}$$

from Eq. (4.20) and (4.36). Assuming that the system is at rest for $t = 0$ it follows that the initial condition vanishes: $\bar{y}_i(\mu) = 0$. With the knowledge of the boundary terms of Eqs. (4.39) and (4.67) Eq. (5.2) can be rewritten as

$$\bar{Y}(\mu, s) = \frac{1}{s^2 - s_\mu^2} \bar{F}_\mathrm{e}(\mu, s). \tag{5.3}$$

This inverse transformation of Eq. (5.1) is represented by a summation over all eigenfunctions of μ and results in the entire solution. The parameters κ_x and κ_y represent the indices of partial solutions. When that transformation is for instance written as

$$\boldsymbol{Y}(\boldsymbol{x}, s) = \sum_{\kappa_x} \sum_{\kappa_y} \frac{1}{N_\mu} \boldsymbol{K}(\boldsymbol{x}, \kappa_x, \kappa_y) \bar{Y}(\mu, s), \tag{5.4}$$

the equation can be interpreted as an double summation of specific partial solutions that are indexed by κ_x and κ_y. Each parameter $\mu(\kappa_x, \kappa_y)$ is hereby allocated to a specific combination of κ_x and κ_y. However the structure of the allocation of μ to κ_x and κ_y can be chosen arbitrarily, because the summation is commutative. The Eq. (5.4) offers a beneficial solution based on two individual summations over κ_x and κ_y. For further calculation this assumption is beneficial and therefore applied. The scaling factor N_μ of Eq. (4.116) can be reformulated by Eq. (4.103)

$$N_\mu = \frac{D L_x L_y}{4 s_\mu^2} \left[\lambda_x^2 + \lambda_y^2\right]^2 \tag{5.5}$$

and then can be inserted together with defintions of $\boldsymbol{Y}(\boldsymbol{x}, s)$ and $\boldsymbol{K}(\boldsymbol{x}, \mu)$ of Eqs. (4.96) and (4.7) in Eq. (5.4)

$$w(\boldsymbol{x}, s) = \frac{4}{\pi^4 D L_x L_y} \sum_{\kappa_x} \sum_{\kappa_y} \frac{s_\mu^2}{s^2 - s_\mu^2} \frac{\bar{F}_\mathrm{e}(\mu, s)}{\left[\frac{\kappa_x^2}{L_x^2} + \frac{\kappa_y^2}{L_y^2}\right]^2} \sin\left(\kappa_x \frac{\pi}{L_x} x\right) \sin\left(\kappa_y \frac{\pi}{L_y} y\right). \tag{5.6}$$

Note that this representation is a dynamic solution and it has to be reformulated for the comparison with the Navier's static solution. Thus, the excitation function is multiplied with a step function at $t = 0$. This can be interpreted as $\frac{1}{s}$ in the temporal frequency domain. So the excitation can be written as

$$\bar{F}_\mathrm{e}(\mu, s) = \bar{P}_\mathrm{e}(\mu) \frac{1}{s} \tag{5.7}$$

and the resulting filtering follows in Fig. 5.1.

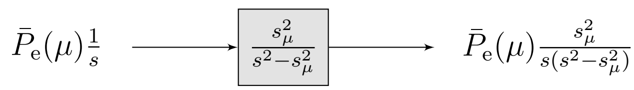


Figure 5.1: The resulting filter with the step function as additional input for the static consideration.

This solution can be further developed by the partial fraction expansion

$$\bar{P}_{\mathrm{e}}(\mu)\frac{s_\mu^2}{s(s^2-s_\mu^2)} = \bar{P}_{\mathrm{e}}(\mu)\left[\frac{A}{s}+\frac{B}{s-s_\mu}+\frac{C}{s+s_\mu}\right] \tag{5.8}$$

$$= \frac{A(s^2-s_\mu^2)+Bs(s+s_\mu)+Cs(s-s_\mu)}{s(s-s_\mu)(s+s_\mu)}. \tag{5.9}$$

with

$$A=-1, \qquad B=\frac{1}{2}, \qquad C=\frac{1}{2}. \tag{5.10}$$

So Eq. (5.8) can be reformulated in the form of

$$\bar{P}_{\mathrm{e}}(\mu)\frac{s_\mu^2}{s(s^2-s_\mu^2)} = \bar{P}_{\mathrm{e}}(\mu)\left[\frac{-1}{s}+\frac{\frac{1}{2}}{s-s_\mu}+\frac{\frac{1}{2}}{s+s_\mu}\right]. \tag{5.11}$$

This resulting term can now be split up in an external and internal part [17]

$$\bar{P}_{\mathrm{e}}(\mu)\frac{s_\mu^2}{s(s^2-s_\mu^2)} = \underbrace{\bar{P}_{\mathrm{e}}(\mu)\left[\frac{-1}{s}\right]}_{External Part}+\underbrace{\bar{P}_{\mathrm{e}}(\mu)\left[\frac{\frac{1}{2}}{s-s_\mu}+\frac{\frac{1}{2}}{s+s_\mu}\right]}_{Internal Part}. \tag{5.12}$$

With regard to the intended comparison of the Navier's static solution just the external part has to be further considered [17, p.306]. The application of the inverse Laplace transformation to the external part then leads to

$$\bar{P}_{\mathrm{e}}(\mu)\left[-\frac{1}{s}\right] \qquad \bullet\!\!-\!\!\circ \qquad \bar{P}_{\mathrm{e}}(\mu)\,[-1]\,\epsilon(t). \tag{5.13}$$

The static case follows for

$$\lim_{t\to\infty}\bar{P}_{\mathrm{e}}(\mu)\,[-1]\,\epsilon(t) = \bar{P}_{\mathrm{e}}(\mu)(-1) \tag{5.14}$$

that furthermore can be reinserted in Eq. (5.6)

$$w(x,y) = \frac{4}{\pi^4 D L_x L_y} \sum_{\kappa_x} \sum_{\kappa_y} \frac{-\bar{P}_\mathrm{e}(\mu)}{\left[\frac{\kappa_x^2}{L_x^2} + \frac{\kappa_y^2}{L_y^2}\right]^2} \sin\left(\kappa_x \frac{\pi}{L_x} x\right) \sin\left(\kappa_y \frac{\pi}{L_y} y\right). \tag{5.15}$$

If the excitation function is now defined in the form of

$$-\bar{P}_\mathrm{e}(\mu) = \int_0^{L_y} \int_0^{L_x} p(x,y) \sin\left(\frac{\kappa_x \pi x}{L_x}\right) \sin\left(\frac{\kappa_y \pi y}{L_y}\right) \mathrm{d}x, \tag{5.16}$$

the direct comparison to the Navier's solution of Eq. (2.161) is possible

$$w(x,y) = \frac{1}{\pi^4 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_{mn}}{\left[\frac{m^2}{L_x^2} + \frac{n^2}{L_y^2}\right]^2} \sin\left(\frac{m \pi x}{L_x}\right) \sin\left(\frac{n \pi y}{L_y}\right) \tag{5.17}$$

with the coeffiecient p_{mn}

$$p_{mn} = \frac{4}{L_x L_y} \int_0^{L_y} \int_0^{L_x} p(x,y) \sin\left(\frac{m \pi x}{L_x}\right) \sin\left(\frac{n \pi y}{L_y}\right) \mathrm{d}x \mathrm{d}y. \tag{5.18}$$

The comparison of the Eqs. (5.15) and (5.17) requires a substitution of the wavenumbers like $\kappa_x = m$ and $\kappa_y = n$. The superposition of two sin-functions with the spatial frequencies of $\lambda_x = \kappa_x \frac{\pi}{L_x}$ and $\lambda_y = \kappa_y \frac{\pi}{L_y}$ are the same and also the scaling factor equals the weighting of Navier's load.

With regard to this comparison it can be stated that the solution (see Eq. (5.6)) of the applied FTM to the PDE for vibrating plates has been shown to correspond to the certain scenario of simple boundary conditions and a static consideration. This means that the general solution of Eq. (5.6) leads to a correct solution for the shown specific case. Due to the fact that the solution of the FTM is a dynamic even more scenarios can be described and have to be tested and proofed in further works.

Chapter 6

Summary and Outlook

The final chapter provides a resume of the thesis and gives a perspective for further works to the topic.

6.1 Summary of the Results

This thesis dealt with the derivation of a partial differential equation based on the physics of vibrating plates, its boundary conditions and several methods for solving this PDE with simple boundary conditions. At first this thesis described the physical concept of vibrating plates in Chapter 2. Beneficial assumptions according to Kirchhoff were introduced and the effect of stresses and strains were elaborated. By means of Hooke's Law the latter concepts were combined and the condition of equilibrium was defined. Based on this plate theory a static and dynamic partial differential equation of the plate were established and possible boundary conditions were discussed. Furthermore, analytical solutions for the partial differential equation were shown and served for later comparisons. In Chapter 3 the Functional Transformation Method was stated in more detail. Within the description of the several steps of the FTM the Laplace and Sturm-Liouville Transformation were explained. Albeit the kernel functions are known for the Laplace Transformation, the kernel function for SLT is unknown and in general dependend on the corresponding partial differential equation. A general approach for the determination of the kernel functions and eigenfrequencies were shown and an inverse transformation for a discrete synthesis algorithm was developed. With regard to the already established physical theory of vibrating plates the Functional Transformation Method was applied to the PDE of plates for simply supported edges in Chapter 4. The determination of the boundary terms were hereby nontrivial and had to be considered in more detail. Solving the eigenvalue problem led then to kernel functions and dispersion relations of the primal and adjoint kernel. In Chapter 5 the calculated results of the FTM were compared to the already existing solution of Navier of Section 2.2.3. To this end, a static consideration of the solution of the FTM had to be developed. Under certain assumptions it was shown that the kernel functions and the scaling factor of the FTM are correct.

6.2 Outlook

For the real-time application of the vibrating plate a discretization of Eq. (5.6) is necessary in order to run a simulation for instance in Matlab. A possible discretization is offered by the impulse invariant transformation of Sec. 3.6. Since in Chapter 5 just the static case was proven, further comparisons for the dynamic consideration have to be performed. Additionally, as this thesis provides the basis for the FTM for vibrating plates with simple boundary conditions the application of complex boundary conditions can be further developed. Therefore, the results of this thesis can be reused, since the transition to complex boundary conditions can be treated separately [18]. An elaborated approach of adjustable boundary conditions was exhibited in papers as in [19][20][21]. The core synthesis of the FTM can be performed with simple boundary conditions, whereas the complex boundary conditions can be applied by an additional feedback loop between outputs and inputs of the transfer function model. The outputs at the edge can be understood as observation and the inputs as boundary excitations. That consideration extends the system to an open loop system for the simple boundary value problem and an closed loop system for the complex boundary value problem.

The new eigenvalues of the complex boundary value problem have not be calculated explicitly, since its effect is represented by a feedback loop. This approach is investigated quite well for one-dimensional PDEs, but not for two-dimensional equations like the plate equation [18].

6.3 Zusammenfassung und Ausblick

Die vorliegende Arbeit beschäftigt sich mit der Herleitung der partiellen Differentialgleichung für schwingende Platten und deren Randbedingungen sowie mit mehreren Methoden zur Lösung dieser partiellen Differentialgleichung. Im Kapitel 2 wurden hierfür die physikalischen Prinzipien einer schwingenden Platte erläutert und dementsprechend vorteilhafte Annahmen nach Kirchhoff getroffen, um die physikalischen Effekte von Druck und Deformation näher zu beschreiben. Diese Konzepte wurden unter Verwendung des Hookschen Gesetzes verknüpft, und der Gleichgewichtszustand wurde definiert. Mithilfe dieser Plattentheorie konnte eine statische und dynamische partielle Differentialgleichung für Plattenschwingungen entwickelt werden. Darauf aufbauend wurden analytische Lösungen aufgezeigt, die zum Vergleich von weiteren Ergebnissen genutzt werden konnten. Im Kapitel 3 wurden die theoretischen Konzepte der funktionalen Transformationsmethode näher erläutert, die anschließend im Kapitel 4 für die partielle Differentialgleichung einer Platte mit einfachen Randbedingungen angewendet wurde. Hierbei war die Berechnung der sich ergebenden Randterme nicht trivial und wurde daher näher untersucht. Die Lösung des Eigenwertproblems führte weiterhin zu den Kernfunktionen und der Dispersionsrelation für den primalen und adjungierten Kern. Im Kapitel 5 wurden diese Ergebnisse mit einer bestehenden mathematischen Lösung von Navier für einfache Randbedingungen verglichen. Dabei konnte gezeigt werden, dass das Ergebnis der funktionalen Transformationsmethode zu einer richtigen Lösung im statischen Fall führt.

Weiterhin können die Resultate für eine Echtzeit-Implementation zum Beispiel in Matlab genutzt werden. Hier wird eine Diskretisierung der obigen Ergebnisse notwendig.

Im Abschnitt 3.6 ist die Theorie der Impulsinvarianz-Transformation erläutert, die hierfür angewendet werden könnte. Des Weiteren dient die vorliegende Arbeit als Grundlage für die Anwendung von komplexen Randbedingungen, da der Übergang zu diesen separat berechnet werden kann [18]. Mögliche Vorgehensweisen zur Anwendung komplexer Randbedingungen sind in folgenden Ausarbeitungen zu finden [19][20][21]. Diese Erweiterung wurde für eindimensionale partielle Differentialgleichungen näher untersucht, jedoch besteht für zweidimensionale Gleichungen, die nicht separiert werden können wie beispielsweise die Plattengleichung, weiterer Forschungsbedarf [18].

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Languages

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- Basic knowledge in JAVA and Matlab-Programming

Erlangen, 29 June 2017
Manuel Werner

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Appendix A

Further Calculations for the Plate Theory

A.1 Dimensions of the Forces and Moments of the Dynamic Plate

Calculating the dimensions of the right side of Eq. (2.45) leads to

$$\left[(\rho h)\frac{\partial^2 w}{\partial t^2}\right] = \left[\frac{\mathrm{kg}}{\mathrm{m}^3}\mathrm{m}\frac{\mathrm{m}}{\mathrm{s}^2} = \frac{\mathrm{kg}}{\mathrm{ms}^2}\right] \tag{A.1}$$

that equals the dimensions of a potential loading

$$[p] = \left[\frac{\mathrm{N}}{\mathrm{m}^2} = \frac{\mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}^2}}{\mathrm{m}^2} = \frac{\mathrm{kg}}{\mathrm{ms}^2}\right] \tag{A.2}$$

and the derivatives of the vertical shear forces (see 2.1.2)

$$\left[\frac{\partial Q^{(x)}}{\partial x}\right] = \left[\frac{1}{m}\left(\frac{\frac{\mathrm{kgm}}{\mathrm{s}^2}}{\mathrm{m}}\right) = \frac{1}{m}\left(\frac{\mathrm{kg}}{\mathrm{s}^2}\right) = \frac{\mathrm{kg}}{\mathrm{ms}^2}\right] \tag{A.3}$$

of the left side.

Appendix B

Further Calculations for the FTM

B.1 Proof of the Matrix Exponential

The solution of the equation

$$\partial_x \boldsymbol{K}(x) = \boldsymbol{Q}\boldsymbol{K}(x) \tag{B.1}$$

is be stated as

$$\boldsymbol{K}(x) = e^{\boldsymbol{Q}x}\boldsymbol{K}(0), \tag{B.2}$$

where the matrix exponential is defined as

$$e^{\boldsymbol{Q}x} = \sum_{n=0}^{\infty} \frac{1}{n!}(\boldsymbol{Q}x)^n. \tag{B.3}$$

Furthermore it is assumed that

$$\boldsymbol{K}(x) = \sum_{n=0}^{\infty} \frac{1}{n!}(\boldsymbol{Q}x)^n \boldsymbol{K}(0) \tag{B.4}$$

which has to be proofed. Therefore it can be shown that

$$\frac{\partial}{\partial x} e^{\boldsymbol{Q}x} = \frac{\partial}{\partial x} \sum_{n=0}^{\infty} \frac{1}{n!}(\boldsymbol{Q}x)^n = \boldsymbol{Q} \sum_{n=0}^{\infty} \frac{1}{n!} n(\boldsymbol{Q}x)^{n-1} \tag{B.5}$$

$$= \boldsymbol{Q} \sum_{n=1}^{\infty} \frac{1}{(n-1)!}(\boldsymbol{Q}x)^{n-1}. \tag{B.6}$$

Using the substitution of $m = n - 1$ leads to

$$\frac{\partial}{\partial x} e^{\boldsymbol{Q}x} = \boldsymbol{Q} \left[\sum_{m=0}^{\infty} \frac{1}{(m)!} (\boldsymbol{Q}x)^m \right] = \boldsymbol{Q} e^{\boldsymbol{Q}x}. \tag{B.7}$$

Thus it is shown that equation (B.1) holds.

Appendix C

Partial Integration

In the following chapters calculations are done that are needed for the partial integration over the entire volume.

C.1 Derivatives of one Direction

In the beginning the product rule for derivatives is shown

$$\partial_{xx}(u^{\mathrm{H}}v) = (\partial_{xx}u^{\mathrm{H}})v + 2\partial_x u^{\mathrm{H}}\partial_x v + u^{\mathrm{H}}(\partial_{xx}v). \tag{C.1}$$

Integrating over this equation results in

$$\int_0^{L_x} \partial_{xx}(u^{\mathrm{H}}v)\mathrm{d}x = \int_0^{L_x} (\partial_{xx}u^{\mathrm{H}})v\mathrm{d}x + 2\int_0^{L_x} \partial_x u^{\mathrm{H}}\partial_x v\mathrm{d}x + \int_0^{L_x} u^{\mathrm{H}}(\partial_{xx}v)\mathrm{d}x. \tag{C.2}$$

The left side can further be calculated as

$$\int_0^{L_x} \partial_{xx}(u^{\mathrm{H}}v)\mathrm{d}x = \left[\partial_x(u^{\mathrm{H}}v)\right]_0^{L_x}. \tag{C.3}$$

Inserting the boundaries requires a description that uses the variables like

$$u^{\mathrm{H}} = u^{\mathrm{H}}(x,y), \qquad v^{\mathrm{H}} = v^{\mathrm{H}}(x,y). \tag{C.4}$$

Equation (C.3) can be reformulated as

$$\left[\partial_x(u^{\mathrm{H}}(x,y)v(x,y))\right]_0^{L_x} = \partial_x\left[(u^{\mathrm{H}}(L_x,y)v(L_x,y))\right] - \partial_x\left[(u^{\mathrm{H}}(0,y)v(0,y))\right]. \tag{C.5}$$

On this equation the product rule for derivatives can be applied again

$$\partial_x \left[(u^{\mathrm{H}}(L_x,y)v(L_x,y))\right] - \partial_x \left[(u^{\mathrm{H}}(0,y)v(0,y))\right] = \tag{C.6}$$

$$\partial_x u^{\mathrm{H}}(L_x,y)v(L_x,y) + u^{\mathrm{H}}(L_x,y)\partial_x v(L_x,y) - \partial_x u^{\mathrm{H}}(0,y)v(0,y) - u^{\mathrm{H}}(0,y)\partial_x v(0,y). \tag{C.7}$$

Another integration over y leads to

$$\int_0^{L_y}\int_0^{L_x} \partial_{xx}(u^{\mathrm{H}}v)\mathrm{d}x\mathrm{d}y = \int_0^{L_y} \partial_x u^{\mathrm{H}}(L_x,y)v(L_x,y)\mathrm{d}y + \int_0^{L_y} u^{\mathrm{H}}(L_x,y)\partial_x v(L_x,y)\mathrm{d}y \tag{C.8}$$

$$- \int_0^{L_y} \partial_x u^{\mathrm{H}}(0,y)v(0,y)\mathrm{d}y - \int_0^{L_y} u^{\mathrm{H}}(0,y)\partial_x v(0,y)\mathrm{d}y. \tag{C.9}$$

With regard to the Eq. (C.2) if follows

$$\int_0^{L_y} \partial_x u^{\mathrm{H}}(L_x,y)v(L_x,y)\mathrm{d}y + \int_0^{L_y} u^{\mathrm{H}}(L_x,y)\partial_x v(L_x,y)\mathrm{d}y \tag{C.10}$$

$$- \int_0^{L_y} \partial_x u^{\mathrm{H}}(0,y)v(0,y)\mathrm{d}y - \int_0^{L_y} u^{\mathrm{H}}(0,y)\partial_x v(0,y)\mathrm{d}y \tag{C.11}$$

$$= \int_0^{L_x} (\partial_{xx}u^{\mathrm{H}})v\mathrm{d}x + 2\int_0^{L_x} (\partial_x u^{\mathrm{H}})(\partial_x v)\mathrm{d}x + \int_0^{L_x} u^{\mathrm{H}}(\partial_{xx}v)\mathrm{d}x. \tag{C.12}$$

Moreover it can be shown that

$$\int_0^{L_y}\int_0^{L_x} (\partial_x u^{\mathrm{H}})(\partial_x v)\mathrm{d}x\mathrm{d}y = \int_0^{L_y} \left[(\partial_x u^{\mathrm{H}})(\partial_x v)\right]_0^{L_x} \mathrm{d}y - \int_0^{L_y}\int_0^{L_x} (\partial_{xx}u^{\mathrm{H}})v\mathrm{d}x\mathrm{d}y \tag{C.13}$$

and with $u^{\mathrm{H}} = u^{\mathrm{H}}(x,y)$ and $v^{\mathrm{H}} = v^{\mathrm{H}}(x,y)$

$$\int_0^{L_y}\int_0^{L_x} (\partial_x u^{\mathrm{H}})(\partial_x v)\mathrm{d}x\mathrm{d}y = \tag{C.14}$$

$$\int_0^{L_y} \left[\partial u^{\mathrm{H}}(L_x,y)v(L_x,y) - \partial u^{\mathrm{H}}(0,y)v(0,y)\right] \mathrm{d}y - \int_0^{L_y}\int_0^{L_x} (\partial_{xx}u^{\mathrm{H}})v\mathrm{d}x\mathrm{d}y. \tag{C.15}$$

Now the Eq. (C.12) can be rewritten as

$$\int_0^{L_y} \partial_x u^{\mathrm{H}}(L_x, y)v(L_x, y)\mathrm{d}y + \int_0^{L_y} u^{\mathrm{H}}(L_x, y)\partial_x v(L_x, y)\mathrm{d}y \tag{C.16}$$

$$-\int_0^{L_y} \partial_x u^{\mathrm{H}}(0, y)v(0, y)\mathrm{d}y - \int_0^{L_y} u^{\mathrm{H}}(0, y)\partial_x v(0, y)\mathrm{d}y \tag{C.17}$$

$$= \int_0^{L_y}\int_0^{L_x} (\partial_{xx} u^{\mathrm{H}})v\mathrm{d}x\mathrm{d}y + 2\int_0^{L_y} \partial_x u^{\mathrm{H}}(L_x, y)v(L_x, y)\mathrm{d}y - 2\int_0^{L_y} \partial_x u^{\mathrm{H}}(0, y)v(0, y)\mathrm{d}y \tag{C.18}$$

$$-2\int_0^{L_y}\int_0^{L_x} (\partial_x u^{\mathrm{H}})v\mathrm{d}x\mathrm{d}y + \int_0^{L_y}\int_0^{L_x} u^{\mathrm{H}}(\partial_{xx} v)\mathrm{d}x\mathrm{d}y. \tag{C.19}$$

This can be further simplified to the following form

$$\langle \partial_{xx} v, u\rangle = \langle v, \partial_{xx} u\rangle - \int_0^{L_y} \partial_x u^{\mathrm{H}}(L_x, y)v(L_x, y)\mathrm{d}y \tag{C.20}$$

$$+\int_0^{L_y} \partial_x u^{\mathrm{H}}(0, y)v(0, y)\mathrm{d}y \tag{C.21}$$

$$+\int_0^{L_y} u^{\mathrm{H}}(L_x, y)\partial_x v(L_x, y)\mathrm{d}y \tag{C.22}$$

$$-\int_0^{L_y} u^{\mathrm{H}}(0, y)\partial_x v(0, y)\mathrm{d}y. \tag{C.23}$$

In the case of two derivatives of y the following equation can be analogously derived

$$\langle \partial_{yy} v, u\rangle = \langle v, \partial_{yy} u\rangle - \int_0^{L_x} \partial_y u^{\mathrm{H}}(x, L_y)v(x, L_y)\mathrm{d}x \tag{C.24}$$

$$+\int_0^{L_x} \partial_y u^{\mathrm{H}}(x, 0)v(x, 0)\mathrm{d}x \tag{C.25}$$

$$+\int_0^{L_x} u^{\mathrm{H}}(x, L_y)\partial_y v(x, L_y)\mathrm{d}x \tag{C.26}$$

$$-\int_0^{L_x} u^{\mathrm{H}}(x, 0)\partial_y v(x, 0)\mathrm{d}x. \tag{C.27}$$

C.2 Derivatives of two Directions

In a similar manner derivatives of x and y can be developed like

$$\partial_x\partial_y(u^{\mathrm{H}}v) = (\partial_x\partial_y u^{\mathrm{H}})v + (\partial_x u^{\mathrm{H}})()\partial_y v) + (\partial_y u^{\mathrm{H}})(\partial_x v) + u^{\mathrm{H}}(\partial_x\partial_y v). \tag{C.28}$$

Integration over x leads to

$$\int_0^{L_x} \partial_x \partial_y (u^{\mathrm{H}} v) \mathrm{d}x = \tag{C.29}$$

$$\int_0^{L_x} (\partial_x \partial_y u^{\mathrm{H}}) v \mathrm{d}x + \int_0^{L_x} (\partial_x u^{\mathrm{H}})(\partial_y v) + \int_0^{L_x} (\partial_y u^{\mathrm{H}})(\partial_x v) \mathrm{d}x + \int_0^{L_x} u^{\mathrm{H}} (\partial_x \partial_y v) \mathrm{d}x. \tag{C.30}$$

The left side of this equation can be further developed to

$$\int_0^{L_x} \partial_x \partial_y (u^{\mathrm{H}} v) \mathrm{d}x = \left[\partial_y (u^{\mathrm{H}} v) \right]_0^{L_x} . \tag{C.31}$$

With $u^{\mathrm{H}} = u^{\mathrm{H}}(x, y)$ and $v^{\mathrm{H}} = v^{\mathrm{H}}(x, y)$ this can be written as

$$\left[\partial_y (u^{\mathrm{H}}(x, y) v(x, y) \right]_0^{L_x} = \partial_y \left[(u^{\mathrm{H}}(L_x, y) v(L_x, y)) \right] - \partial_y \left[(u^{\mathrm{H}}(0, y) v(0, y)) \right] . \tag{C.32}$$

The integration over y leads to

$$\int_0^{L_y} \int_0^{L_x} \partial_x \partial_y (u^{\mathrm{H}} v) \mathrm{d}x \mathrm{d}y = \int_0^{L_y} \partial_y \left[u^{\mathrm{H}}(L_x, y) v(L_x, y) \right] \mathrm{d}y - \int_0^{L_y} \partial_y \left[u^{\mathrm{H}}(0, y) v(0, y) \right] \mathrm{d}y \tag{C.33}$$

$$= \left[u^{\mathrm{H}}(L_x, y) v(L_x, y) \right]_0^{L_y} - \left[u^{\mathrm{H}}(0, y) v(0, y) \right]_0^{L_y} \tag{C.34}$$

$$= \left[u^{\mathrm{H}}(L_x, L_y) v(L_x, L_y) \right] - \left[u^{\mathrm{H}}(L_x, 0) v(L_x, 0) \right] \tag{C.35}$$

$$- \left[u^{\mathrm{H}}(0, L_y) v(0, L_y) \right] + \left[u^{\mathrm{H}}(0, 0) v(0, 0) \right] . \tag{C.36}$$

Moreover the second and third integral of the right side of the Eq. (C.30) can be solved like

$$\int_0^{L_y} \int_0^{L_x} (\partial_x u^{\mathrm{H}})(\partial_y v) \mathrm{d}x \mathrm{d}y = \int_0^{L_y} \left[u^{\mathrm{H}} (\partial_y v) \right]_0^{L_x} \mathrm{d}y - \int_0^{L_y} \int_0^{L_x} u^{\mathrm{H}} (\partial_x \partial_y v) \mathrm{d}x \mathrm{d}y \tag{C.37}$$

$$= \int_0^{L_y} \left[u^{\mathrm{H}}(L_x, y) \partial_y v(L_x, y) - u^{\mathrm{H}}(0, y) \partial_y v(0, y) \right] \mathrm{d}y - \int_0^{L_y} \int_0^{L_x} u^{\mathrm{H}} (\partial_x \partial_y v) \mathrm{d}x \mathrm{d}y. \tag{C.38}$$

Analogously, the following calculations can be derived

$$\int_0^{L_y}\int_0^{L_x}(\partial_y u^{\mathrm{H}})(\partial_x v)\mathrm{d}x\mathrm{d}y = \tag{C.39}$$

$$\int_0^{L_x}\int_0^{L_y}(\partial_y u^{\mathrm{H}})(\partial_x v)\mathrm{d}y\mathrm{d}x = \int_0^{L_x}\left[u^{\mathrm{H}}(\partial_x v)\right]_0^{L_y}\mathrm{d}x - \int_0^{L_x}\int_0^{L_y}u^{\mathrm{H}}(\partial_x\partial_y v)\mathrm{d}y\mathrm{d}x \tag{C.40}$$

$$= \int_0^{L_x}\left[u^{\mathrm{H}}(x,L_y)\partial_x v(x,L_y) - u^{\mathrm{H}}(x,0)\partial_x v(x,0)\right]\mathrm{d}y - \int_0^{L_x}\int_0^{L_y}u^{\mathrm{H}}(\partial_x\partial_y v)\mathrm{d}y\mathrm{d}x. \tag{C.41}$$

Inserting Eqs. (C.38) and (C.41) into Eq. (C.30) and (C.36) leads to

$$\langle(\partial_{xy}v),u\rangle = \langle v,(\partial_{xy}u)\rangle \tag{C.42}$$

$$+\int_0^{L_y}\left[u^{\mathrm{H}}(L_x,y)\partial_y v(L_x,y) - u^{\mathrm{H}}(0,y)\partial_y v(0,y)\right]\mathrm{d}y \tag{C.43}$$

$$+\int_0^{L_x}\left[u^{\mathrm{H}}(x,L_y)\partial_x v(x,L_y) - u^{\mathrm{H}}(x,0)\partial_x v(x,0)\right]\mathrm{d}y \tag{C.44}$$

$$-\left[u^{\mathrm{H}}(L_x,L_y)v(L_x,L_y)\right] + \left[u^{\mathrm{H}}(L_x,0)v(L_x,0)\right] \tag{C.45}$$

$$+\left[u^{\mathrm{H}}(0,L_y)v(0,L_y)\right] - \left[u^{\mathrm{H}}(0,0)v(0,0)\right]. \tag{C.46}$$

Appendix D

Proof of the Kernel Functions Using The Eigenvalue Problem

D.1 Dimension of the Eigenvalue

Eq. (4.103) defines the following dispersion relation

$$s_\mu^2 = \frac{-D}{\rho h}\left[\lambda_x^2 + \lambda_y^2\right]^2. \tag{D.1}$$

With $[D] = \mathrm{kg}\frac{\mathrm{m}^2}{\mathrm{s}^2}$ it follows

$$[s_\mu^2] = \frac{\mathrm{kg}\frac{\mathrm{m}^2}{\mathrm{s}^2}}{\frac{\mathrm{kg}}{\mathrm{m}^3}\mathrm{m}}\frac{1}{\mathrm{m}^4} = \frac{\mathrm{m}^4}{\mathrm{s}^2}\frac{1}{\mathrm{m}^4} = \frac{1}{\mathrm{s}^2}. \tag{D.2}$$

D.2 Proof of the Kernel Functions Using The Eigenvalue Problem

In order to proof the definition of primal and adjoint kernel functions its result can be inserted in the eigenvalue problem. The primal eigenvalue problem can be formulated as

$$L\boldsymbol{K}(\boldsymbol{x},\mu) = s_\mu^2 \boldsymbol{C}\boldsymbol{K}(\boldsymbol{x},\mu), \qquad \boldsymbol{x} \in V. \tag{D.3}$$

The left side can be calculated by using the differential operator L from Eq. (4.11) and the kernel from Eq. (4.96)

$$\boldsymbol{L}\boldsymbol{K}(\boldsymbol{x},\mu) = \begin{bmatrix} D(\lambda_x^2+\nu\lambda_y^2)\lambda_x^2K_1(\boldsymbol{x},\mu) + D(\lambda_y^2+\nu\lambda_x^2)\lambda_y^2K_1 + 2D(1-\nu)\lambda_x^2\lambda_y^2K_1(\boldsymbol{x},\mu) \\ -D(\partial_{xx}+\nu\partial_{yy})K_1(\boldsymbol{x},\mu) - D(\lambda_x^2+\nu\lambda_y^2)K_1(\boldsymbol{x},\mu) \\ -D(\partial_{yy}+\nu\partial_{xx})K_1(\boldsymbol{x},\mu) - D(\lambda_y^2+\nu\lambda_x^2)K_1(\boldsymbol{x},\mu) \\ -D(1-\nu)\partial_{xy}K_1(\boldsymbol{x},\mu) + D\lambda_x\lambda_y(1-\nu)\cos(\lambda_x x)\cos(\lambda_y y) \end{bmatrix}$$

$$= \begin{bmatrix} DK_1(\boldsymbol{x},\mu)(\lambda_x^2+\lambda_y^2)^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{D.4}$$

The right side of Eq. (D.3) can be calculated by reusing s_μ^2 from Eq. (4.103) and $\boldsymbol{C}$ from Eq. (4.9)

$$s_\mu^2\boldsymbol{C}\boldsymbol{K}(\boldsymbol{x},\mu) = \begin{bmatrix} DK_1(\boldsymbol{x},\mu)(\lambda_x^2+\lambda_y^2)^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{D.5}$$

that equals the result from Eq. (D.4). A similar proof can be shown for the adjoint eigenvalue problem

$$\tilde{L}\tilde{\boldsymbol{K}}(\boldsymbol{x},\mu) = s_\mu^{*2}\boldsymbol{C}^{\mathrm{H}}\tilde{\boldsymbol{K}}(\boldsymbol{x},\mu), \qquad \boldsymbol{x} \in V. \tag{D.6}$$

The adjoint differential operator is known from Eq. (4.79) and the adjoint kernel from Eq. (4.101). Based on that it follows

$$\begin{aligned}
\tilde{L}\tilde{\boldsymbol{K}}(\boldsymbol{x},\mu) &= \begin{bmatrix} -D(\partial_{xx}+\nu\partial_{yy})\lambda_x^2\tilde{K}_1 - D(\partial_{yy}+\nu\partial_{xx})\lambda_y^2\tilde{K}_1(\boldsymbol{x},\mu) + 2\lambda_x\lambda_y D(1-\nu)\partial_{xy}\tilde{K}_1(\boldsymbol{x},\mu) \\ -\partial_{xx}\tilde{K}_1(\boldsymbol{x},\mu) - \lambda_x^2\tilde{K}_1(\boldsymbol{x},\mu) \\ -\partial_{yy}\tilde{K}_1(\boldsymbol{x},\mu) - \lambda_y^2\tilde{K}_1(\boldsymbol{x},\mu) \\ -2\partial_{xy}\tilde{K}_1(\boldsymbol{x},\mu) + -2\lambda_x\lambda_y\left[cos(\lambda_x x)cos(\lambda_y y)\right] \end{bmatrix} \\
&= \begin{bmatrix} -D\lambda_x^2(-\lambda_x^2-\nu\lambda_y^2)\tilde{K}_1(\boldsymbol{x},\mu) - D\lambda_y^2(-\lambda_y^2-\nu\lambda_x^2)\tilde{K}_1(\boldsymbol{x},\mu) + 2\lambda_x^2\lambda_y^2 D(1-\nu)\tilde{K}_1(\boldsymbol{x},\mu) \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} D\tilde{K}_1(\boldsymbol{x},\mu)\left[\lambda_x^2+\lambda_y^2\right]^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad \text{(D.7)}
\end{aligned}$$

Inserting the results from Eqs. (4.105) and (4.82) leads to

$$s_\mu^{*2}\boldsymbol{C}^{\mathrm{H}}\tilde{\boldsymbol{K}}(\boldsymbol{x},\mu) = \begin{bmatrix} -s_\mu^{*2}\rho h\tilde{K}_1(\boldsymbol{x},\mu) \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} D\left[\lambda_x^2+\lambda_y^2\right]^2\tilde{K}_1(\boldsymbol{x},\mu) \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad \text{(D.8)}$$

that equals Eq. (D.7). Thus the adjoint kernel functions are valid solutions with regard to its eigenvalue problem.

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